

## UNITARY OPERATORS: ACTIVE AND PASSIVE TRANSFORMATIONS OF AN OPERATOR

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.7.1 - 1.7.2.

A unitary operator transforms one orthonormal basis to another. Therefore if we have an operator  $\Omega$  with matrix elements  $\Omega_{ij}(\{f\}) = \langle f_i | \Omega | f_j \rangle$  in one orthonormal basis  $(f_1, \dots, f_n)$ , we can transform the basis to another orthonormal basis  $(e_1, \dots, e_n)$  by a unitary transformation  $U$  so that

$$(0.1) \quad |e_i\rangle = U |f_i\rangle$$

This results in a transformation of the operator  $\Omega$ 's matrix elements:

$$(0.2) \quad \Omega_{ij}(\{e\}) = \langle e_i | \Omega | e_j \rangle$$

$$(0.3) \quad = \langle U f_i | \Omega | U f_j \rangle$$

$$(0.4) \quad = \langle f_i | U^\dagger \Omega U | f_j \rangle$$

Thus we can view the transformation as either a transformation of the basis vectors  $e_i = U f_i$ , known as an *active transformation*, or as a transformation of the operator according to  $\Omega \rightarrow U^\dagger \Omega U$ , known as a *passive transformation*. The matrix elements of  $U^\dagger \Omega U$  in the original basis  $\{f\}$  are equal to the matrix elements of the original operator  $\Omega$  in the new basis  $\{e\}$ .

We've already seen a few results about the trace and determinant of products of matrices. We'll list these here for reference:

- $\text{Tr}(\Omega\Lambda) = \text{Tr}(\Lambda\Omega)$ . That is, even if the operators don't commute, the trace of a product of operators doesn't depend on the order of the operators in the product.
- The trace of a product of 3 or more operators is invariant under cyclic permutation.  $\text{Tr}(\Omega\Lambda\theta) = \text{Tr}(\Lambda\theta\Omega) = \text{Tr}(\theta\Omega\Lambda)$ . This follows directly from the previous result. For example, we can define  $A \equiv \Lambda\theta$  so that  $\text{Tr}(\Omega\Lambda\theta) = \text{Tr}(\Omega A) = \text{Tr}(A\Omega) = \text{Tr}(\Lambda\theta\Omega)$ .
- The determinant of a unitary matrix is a complex number with modulus 1.

We can use these to prove a couple of further results about unitary transformations.

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The trace of an operator is invariant under a unitary transformation:

$$(0.5) \quad \text{Tr}(U^\dagger \Omega U) = \text{Tr}(U U^\dagger \Omega) = \text{Tr} \Omega$$

since  $U U^\dagger = I$ .

Finally, the determinant of an operator is also invariant under a unitary transformation. Since the determinant of a product is the product of the determinants,

$$(0.6) \quad \det(U^\dagger \Omega U) = \det U^\dagger \det \Omega \det U$$

$$(0.7) \quad = e^{-i\alpha} \det \Omega e^{i\alpha}$$

$$(0.8) \quad = \det \Omega$$

In the second line we used the fact that  $\det U$  is a complex number with unit modulus, and the fact that  $\det U^\dagger = (\det U)^*$ .

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