

## EIGENVALUES AND EIGENVECTORS - EXAMPLES

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.8.1 - 1.8.4.

Here are a few examples of calculating eigenvalues and eigenvectors.

**Example 1.** Find the eigenvalues and normalized eigenvectors of

$$(0.1) \quad \Omega = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

The eigenvalues are solutions of  $\det(\Omega - \lambda I) = 0$  which gives, calculating the determinant down the first column:

$$(0.2) \quad (1 - \lambda)(2 - \lambda)(4 - \lambda) = 0$$

$$(0.3) \quad \lambda = 1, 2, 4$$

The eigenvectors  $v_i$  satisfy  $(\Omega - \lambda_i I)v_i = 0v_i$  for each eigenvalue  $\lambda_i$ . We get, for  $\lambda_1 = 1$ :

$$(0.4) \quad \begin{bmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving, we find

$$(0.5) \quad b = c = 0$$

$$(0.6) \quad a = \text{anything}$$

Thus a normalized eigenvector is

$$(0.7) \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For  $\lambda_2 = 2$ , we have

$$(0.8) \quad \begin{bmatrix} -1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving:

$$(0.9) \quad b = -2c$$

$$(0.10) \quad a = 3b + c$$

$$(0.11) \quad = -5c$$

Choosing  $c = 1$  and normalizing, we have

$$(0.12) \quad v_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

Finally, for  $\lambda_3 = 4$  we have

$$(0.13) \quad \begin{bmatrix} -3 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving:

$$(0.14) \quad b = 0$$

$$(0.15) \quad 3a = 3b + c$$

$$(0.16) \quad = c$$

Choosing  $a = 1$  and normalizing:

$$(0.17) \quad v_3 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

The matrix  $\Omega$  is not Hermitian since  $\Omega^\dagger \neq \Omega$ , and we can see by inspection that the eigenvectors are not orthogonal.

**Example 2.** Now we have

$$(0.18) \quad \Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

It is hermitian since  $\Omega^\dagger = \Omega^T = \Omega$ . The eigenvalues are found from

$$(0.19) \quad (-\lambda)^3 + \lambda = 0$$

$$(0.20) \quad \lambda = 0, -1, 1$$

Solving for the eigenvectors in the same way as in the last example, we get, for  $\lambda_i = 0, -1, 1$  in that order:

$$(0.21) \quad v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$(0.22) \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$(0.23) \quad v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The eigenvectors are orthogonal, as required for a hermitian matrix. We can diagonalize  $\Omega$  by means of a unitary transformation  $U$ , where the columns of  $U$  are the eigenvectors of  $\Omega$ . We have

$$(0.24) \quad U = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$(0.25) \quad U^\dagger = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

We can verify by direct matrix multiplication that

$$(0.26) \quad U^\dagger \Omega U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the order of eigenvalues in the diagonal is determined by the order in which we place the columns in  $U$ .

**Example 3.** We now have the hermitian matrix

$$(0.27) \quad \Omega = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The eigenvectors follow from

$$(0.28) \quad (1 - \lambda) \left[ \left( \frac{3}{2} - \lambda \right)^2 - \frac{1}{4} \right] = 0$$

$$(0.29) \quad \lambda = 1, 1, 2$$

Thus the eigenvalue  $\lambda = 1$  is degenerate. We can find the eigenvector corresponding to  $\lambda_3 = 2$  in the usual way and get

$$(0.30) \quad v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The other two eigenvectors span a 2-d subspace that must be orthogonal to  $v_3$  (since  $\Omega$  is hermitian; in the more general case, the orthogonality is not guaranteed). We can therefore find two vectors  $v_1, v_2$  in the subspace by requiring  $\langle v_1, v_3 \rangle = \langle v_2, v_3 \rangle = 0$ . That is, if

$$(0.31) \quad v_{1,2} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

we must have

$$(0.32) \quad a = \text{anything}$$

$$(0.33) \quad b = c$$

These two equations can be satisfied by a variety of  $v_1$  and  $v_2$ , but if we want  $\langle v_1, v_2 \rangle = 0$  as well, we can choose  $a = 1$  and  $b = c = 1$ , then normalize, to get

$$(0.34) \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(0.35) \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The 2-d subspace spanned by  $v_1$  and  $v_2$  is therefore

$$(0.36) \quad v = av_1 + bv_2 = \begin{bmatrix} a \\ b \\ b \end{bmatrix}$$

Thus any normalized eigenvector of  $\lambda = 1$  has the form

$$(0.37) \quad e = \frac{1}{\sqrt{a^2 + 2b^2}} \begin{bmatrix} a \\ b \\ b \end{bmatrix}$$

**Example 4.** Now let's look at a non-hermitian matrix:

$$(0.38) \quad \Omega = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

The eigenvalues are found from

$$(0.39) \quad (4 - \lambda)(2 - \lambda) + 1 = 0$$

$$(0.40) \quad (\lambda - 3)^2 = 0$$

$$(0.41) \quad \lambda = 3, 3$$

Thus there is one degenerate eigenvalue. To find the eigenvector(s), we solve  $(\Omega - \lambda I)v = 0$  as usual:

$$(0.42) \quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

This gives only one condition, namely  $a = -b$ . Thus there is only one normalized eigenvector:

$$(0.43) \quad v = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$