References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercises 1.8.6 - 1.8.7.

The spectral theorem for states that any normal operator Ω in a complex vector space is unitarily diagonalizable, that is

$$D_M = U^{\dagger}\Omega U \tag{0.1}$$

where U is a unitary operator and D_M is a diagonal matrix, whose diagonal elements are the eigenvalues ω_i of Ω. We can use this to derive a couple of relations about the trace and determinant of normal operators. Remember that hermitian and unitary operators are both normal.

Since the determinant is invariant under a unitary transformation, we have

$$\begin{aligned}
\det D_M &= \det\left(U^{\dagger}\Omega U\right) && (0.2)\\
&= \det U^{\dagger}\det\Omega\det U && (0.3)\\
&= e^{-i\alpha}\times\det\Omega\times e^{i\alpha} && (0.4)\\
&= \det\Omega && (0.5)
\end{aligned}$$

where we've used the facts that the determinant of a product is the product of the determinants, and the determinant of a unitary matrix is a complex number $e^{i\alpha}$ with unit modulus. Since the determinant of a diagonal matrix is the product of its diagonal elements, we see that for a normal matrix, its determinant is the product of its eigenvalues:

$$\det\Omega = \prod_i \omega_i \tag{0.6}$$

The trace of a product is equal to the trace of a cyclic permutation of that product, so we have

$$\begin{aligned}
\mathrm{Tr}D_M &= \mathrm{Tr}\left(U^{\dagger}\Omega U\right) && (0.7)\\
&= \mathrm{Tr}\left(UU^{\dagger}\Omega\right) && (0.8)\\
&= \mathrm{Tr}\Omega && (0.9)
\end{aligned}$$

Therefore, the trace of a normal operator is the sum of its eigenvalues:

$$\mathrm{Tr}\Omega = \sum_i \omega_i \tag{0.10}$$

We can use these two results as an alternative way to calculate the eigenvalues of a normal matrix. For example, suppose

$$\Omega=\left[\begin{array}{cc} 1 & 2\\ 2 & 1 \end{array}\right] \tag{0.11}$$

We have

$$\det\Omega = -3=\omega_{1}\omega_{2} \tag{0.12}$$
$$\mathrm{Tr}\Omega = 2=\omega_{1}+\omega_{2} \tag{0.13}$$

Solving these two equations gives

$$-3 = \left(2-\omega_{2}\right)\omega_{2} \tag{0.14}$$
$$\omega = -1,3 \tag{0.15}$$

We can also calculate them using the old determinant formula $\det\left(\Omega-\omega I\right)=0$:

$$\left(1-\omega\right)^{2}-4 = 0 \tag{0.16}$$
$$\omega = -1,3 \tag{0.17}$$

PINGBACKS

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