ANGULAR MOMENTUM AS AN EIGENVECTOR PROBLEM

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercise 1.8.9.

The angular momentum in classical mechanics of a collection of point masses m_a located at positions $\mathbf{r}_a$ and moving with a common angular velocity ω about a common axis is given by

$$\mathbf{L}=\sum_a m_a\left(\mathbf{r}_a\times\mathbf{v}_a\right) \tag{1}$$

where $\mathbf{v}_a=\omega\times\mathbf{r}_a$ is the linear velocity of m_a. We can use the vector identity

$$\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\mathbf{B}\left(\mathbf{A}\cdot\mathbf{C}\right)-\mathbf{C}\left(\mathbf{A}\cdot\mathbf{B}\right)$$

to write

$$\begin{aligned}\mathbf{r}_a\times\mathbf{v}_a &= \mathbf{r}_a\times\left(\omega\times\mathbf{r}_a\right) && (2)\\ &= r_a^2\omega-\mathbf{r}_a\left(\mathbf{r}_a\cdot\omega\right) && (3)\end{aligned}$$

In terms of components, this is

$$\begin{aligned}\left[\mathbf{r}_a\times\mathbf{v}_a\right]_i &= r_a^2\omega_i-\left(r_a\right)_i\sum_j\left(r_a\right)_j\omega_j && (4)\\ &=\sum_j\left[r_a^2\omega_j\delta_{ij}-\left(r_a\right)_i\left(r_a\right)_j\omega_j\right] && (5)\\ &=\sum_j\left[r_a^2\delta_{ij}-\left(r_a\right)_i\left(r_a\right)_j\right]\omega_j && (6)\end{aligned}$$

We can therefore write the angular momentum as

$$\begin{aligned}\mathbf{L}_i &= \sum_j\sum_a m_a\left[r_a^2\delta_{ij}-\left(r_a\right)_i\left(r_a\right)_j\right]\omega_j && (7)\\ &\equiv\sum_j M_{ij}\omega_j && (8)\end{aligned}$$

where the matrix M is

$$M_{ij} \equiv \sum_a m_a \left[r_a^2 \delta_{ij} - (r_a)_i (r_a)_j \right] \tag{9}$$

From the definition, we see that M is real and symmetric (interchanging i and j shows that $M_{ij} = M_{ji}$), so M is hermitian.

In Dirac's notation, we have the matrix equation

$$|L\rangle = M |\omega\rangle \tag{10}$$

From this equation, we can see that $\mathbf{L}$ and ω are parallel only if ω is an eigenvector of M. If the eigenvalues of M are non-degenerate, there are therefore three directions for ω such that $\mathbf{L}$ and ω are parallel, and these directions can be found by solving for the eigenvectors of M.

If some of the eigenvalues are degenerate, then there is a range of directions over which $\mathbf{L}$ and ω can be parallel. In the case of a sphere, all 3 eigenvalues of M must be the same, as all directions are axes of symmetry of the sphere.

As an example, suppose we have only one mass $m = 1$ with position

$$\mathbf{r} = [1, 1, 0] \tag{11}$$

We can work out M by substituting into 9:

$$M = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{12}$$

The eigenvalues are 0, 2 and 2 with corresponding eigenvectors

$$|\lambda = 0\rangle = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{13}$$

$$|\lambda = 2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{14}$$

Thus if ω is a linear combination of the two eigenvectors for $\lambda = 2$, it will be parallel to $\mathbf{L}$. If ω is parallel to $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{L} = 0$, as in this case ω is parallel to $\mathbf{r}$ so $\omega \times \mathbf{r} = 0$, and the mass is located on the axis of rotation so has no angular momentum.