ANGULAR MOMENTUM AS AN EIGENVECTOR PROBLEM

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The angular momentum in classical mechanics of a collection of point masses $m_a$ located at positions $r_a$ and moving with a common angular velocity $\omega$ about a common axis is given by

$$L = \sum_{a} m_a (r_a \times v_a)$$  \hspace{1cm} (1)

where $v_a = \omega \times r_a$ is the linear velocity of $m_a$. We can use the vector identity

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

to write

$$r_a \times v_a = r_a \times (\omega \times r_a)$$ \hspace{1cm} (2)

$$= r_a^2 \omega - r_a (r_a \cdot \omega)$$ \hspace{1cm} (3)

In terms of components, this is

$$[r_a \times v_a]_i = r_a^2 \omega_i - (r_a)_i \sum_j (r_a)_j \omega_j$$ \hspace{1cm} (4)

$$= \sum_j \left[ r_a^2 \omega_j \delta_{ij} - (r_a)_i (r_a)_j \omega_j \right]$$ \hspace{1cm} (5)

$$= \sum_j \left[ r_a^2 \delta_{ij} - (r_a)_i (r_a)_j \right] \omega_j$$ \hspace{1cm} (6)

We can therefore write the angular momentum as

$$L_i = \sum_j \sum_a m_a \left[ r_a^2 \delta_{ij} - (r_a)_i (r_a)_j \right] \omega_j$$ \hspace{1cm} (7)

$$\equiv \sum_j M_{ij} \omega_j$$ \hspace{1cm} (8)

where the matrix $M$ is
\[ M_{ij} \equiv \sum_a m_a \left[ r_a^2 \delta_{ij} - (r_a)_i (r_a)_j \right] \] (9)

From the definition, we see that \( M \) is real and symmetric (interchanging \( i \) and \( j \) shows that \( M_{ij} = M_{ji} \)), so \( M \) is hermitian.

In Dirac’s notation, we have the matrix equation

\[ |L\rangle = M |\omega\rangle \] (10)

From this equation, we can see that \( L \) and \( \omega \) are parallel only if \( \omega \) is an eigenvector of \( M \). If the eigenvalues of \( M \) are non-degenerate, there are therefore three directions for \( \omega \) such that \( L \) and \( \omega \) are parallel, and these directions can be found by solving for the eigenvectors of \( M \).

If some of the eigenvalues are degenerate, then there is a range of directions over which \( L \) and \( \omega \) can be parallel. In the case of a sphere, all 3 eigenvalues of \( M \) must be the same, as all directions are axes of symmetry of the sphere.

As an example, suppose we have only one mass \( m = 1 \) with position

\[ \mathbf{r} = [1, 1, 0] \] (11)

We can work out \( M \) by substituting into (9)

\[ M = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \] (12)

The eigenvalues are 0, 2 and 2 with corresponding eigenvectors

\[ |\lambda = 0\rangle = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \] (13)

\[ |\lambda = 2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \] (14)

Thus if \( \omega \) is a linear combination of the two eigenvectors for \( \lambda = 2 \), it will be parallel to \( L \). If \( \omega \) is parallel to \( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \), \( L = 0 \), as in this case \( \omega \) is parallel to \( \mathbf{r} \) so \( \omega \times \mathbf{r} = 0 \), and the mass is located on the axis of rotation so has no angular momentum.