

## COUPLED MASSES ON SPRINGS - A SOLUTION USING MATRIX DIAGONALIZATION

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercise 1.8.11.

Here's a practical example of how changing the basis by diagonalizing a hermitian matrix can make a problem easier to solve. Suppose we have two identical masses  $m$  free to slide in one dimension on a frictionless horizontal surface. The two masses are connected to 3 springs, with the spring on the left attached to a solid support at one end and to mass #1 at the other, the middle spring connected between the two masses, and the spring on the right connected to mass #2 at one end and to a solid support at the other. The springs all have spring constant  $k$ . Define two coordinates  $x_1$  and  $x_2$  to be the positions of the two masses, with  $x_i = 0$  corresponding to the location at which mass  $i$  is at rest in equilibrium.

Now suppose that the two masses are displaced from their respective equilibrium points, so that  $x_1$  and  $x_2$  are non-zero. The length of the spring to the left of mass 1 is changed (stretched or compressed, depending on the sign of  $x_1$ ) by  $x_1$ , so exerts a force  $F_1 = -kx_1$  on mass 1. The length of the spring in the middle is changed by  $x_2 - x_1$ , so it exerts a force  $F_{12} = k(x_2 - x_1)$  on mass 1, and an equal and opposite force  $F_{21} = -k(x_2 - x_1)$  on mass 2. Finally, the length of the spring on the right is changed by  $x_2$  and exerts a force  $F_2 = -kx_2$  on mass 2. By applying Newton's law  $F = ma$ , we get the set of equations of motion:

$$(0.1) \quad \ddot{x}_1 = -2\frac{k}{m}x_1 + \frac{k}{m}x_2$$

$$(0.2) \quad \ddot{x}_2 = \frac{k}{m}x_1 - 2\frac{k}{m}x_2$$

While it's possible to solve such a coupled system directly, we can see how an easier method can be found by using matrix algebra. The 2 equations above can be written as a matrix equation

$$(0.3) \quad |\ddot{x}(t)\rangle = \Omega |x(t)\rangle$$

If we use the basis in which the displacement of each mass is taken to be independent of the other, we have the two basis vectors

$$(0.4) \quad |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(0.5) \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In this basis

$$(0.6) \quad |x(t)\rangle = x_1(t) |1\rangle + x_2(t) |2\rangle$$

Here, the  $x_i$ s are just numbers; the vector nature of the equation is delegated to the basis vectors.

In this basis,  $\Omega$  is the operator whose matrix form is

$$(0.7) \quad \Omega = \begin{bmatrix} -2\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -2\frac{k}{m} \end{bmatrix}$$

Since  $\Omega$  is hermitian, it can be diagonalized by finding its eigenvalues and normalized eigenvectors, and forming a unitary operator  $U$  whose columns are these eigenvectors. The basis vectors are now these eigenvectors  $|I\rangle$  and  $|II\rangle$  (I'm sticking to Shankar's notation, even though it's a bit clumsy), and they are found from  $|1\rangle$  and  $|2\rangle$  by applying the unitary transformation, that is

$$(0.8) \quad |I\rangle = U |1\rangle$$

$$(0.9) \quad |II\rangle = U |2\rangle$$

These transformations can be inverted:

$$(0.10) \quad |1\rangle = U^\dagger |I\rangle$$

$$(0.11) \quad |2\rangle = U^\dagger |II\rangle$$

Thus we can insert this into 0.3 and use  $UU^\dagger = I$  to get

$$(0.12) \quad U^\dagger |\ddot{x}(t)\rangle = U^\dagger \Omega U U^\dagger |x(t)\rangle$$

and  $U^\dagger \Omega U$  is the diagonalized version of  $\Omega$ .

Shankar goes through the details of the calculation, with the results

$$(0.13) \quad |I\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(0.14) \quad |II\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(0.15) \quad U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$(0.16) \quad U^\dagger \Omega U = \begin{bmatrix} -\omega_1^2 & 0 \\ 0 & -\omega_2^2 \end{bmatrix}$$

where

$$(0.17) \quad \omega_1 = \sqrt{\frac{k}{m}}$$

$$(0.18) \quad \omega_2 = \sqrt{\frac{3k}{m}}$$

Using  $|I\rangle$  and  $|II\rangle$  as the basis, the differential equations become decoupled, and we have

$$(0.19) \quad \ddot{x}_i + \omega_i^2 x_i = 0$$

for  $i = I, II$ .

Second order ODEs require two initial conditions to be fully solved, and here we're assuming that both masses start off at rest, so that  $\dot{x}_i(t) = 0$  for  $i = I, II$ . In this case, the solutions are

$$(0.20) \quad x_i(t) = x_i(0) \cos \omega_i t$$

for  $i = I, II$ .

(A full, general solution would also have a  $\sin \omega_i t$  term, but this disappears because we require  $\dot{x}_i(t) = 0$ .)

The vector solution in the diagonal basis is therefore

$$(0.21) \quad \begin{bmatrix} x_I(t) \\ x_{II}(t) \end{bmatrix} = |I\rangle x_I(0) \cos \omega_I t + |II\rangle x_{II}(0) \cos \omega_{II} t$$

We now need to figure out what the coefficients  $x_I(0)$  and  $x_{II}(0)$  are. Assuming we know the initial position of each mass in the original basis as  $x_1(0)$  and  $x_2(0)$ , we can find  $x_I(0)$  and  $x_{II}(0)$  by projecting  $x_1(0)$  and  $x_2(0)$  onto the basis  $|I\rangle$  and  $|II\rangle$ . That is, we have

$$(0.22) \quad \begin{bmatrix} x_I(0) \\ x_{II}(0) \end{bmatrix} = U \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$(0.23) \quad = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$(0.24) \quad = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1(0) + x_2(0) \\ x_1(0) - x_2(0) \end{bmatrix}$$

We get

$$(0.25)$$

$$(0.26) \quad \begin{bmatrix} x_I(t) \\ x_{II}(t) \end{bmatrix} = \frac{x_1(0) + x_2(0)}{\sqrt{2}} |I\rangle \cos \omega_I t + \frac{x_1(0) - x_2(0)}{\sqrt{2}} |II\rangle \cos \omega_{II} t$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} [x_1(0) + x_2(0)] \cos \sqrt{\frac{k}{m}} t + [x_1(0) - x_2(0)] \cos \sqrt{\frac{3k}{m}} t \\ [x_1(0) + x_2(0)] \cos \sqrt{\frac{k}{m}} t - [x_1(0) - x_2(0)] \cos \sqrt{\frac{3k}{m}} t \end{bmatrix}$$

where in the last line we substituted using 0.13 to write everything in terms of the original basis  $|1\rangle$  and  $|2\rangle$ .

For the special case where the initial positions are given by  $|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , we have  $x_1(0) = 1$  and  $x_2(0) = 0$ , so that

$$(0.27) \quad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \cos \sqrt{\frac{k}{m}} t + \cos \sqrt{\frac{3k}{m}} t \\ \cos \sqrt{\frac{k}{m}} t - \cos \sqrt{\frac{3k}{m}} t \end{bmatrix}$$

Going back to 0.26, we can write the solution as a matrix equation

$$(0.28)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \cos \sqrt{\frac{k}{m}} t + \cos \sqrt{\frac{3k}{m}} t & \cos \sqrt{\frac{k}{m}} t - \cos \sqrt{\frac{3k}{m}} t \\ \cos \sqrt{\frac{k}{m}} t - \cos \sqrt{\frac{3k}{m}} t & \cos \sqrt{\frac{k}{m}} t + \cos \sqrt{\frac{3k}{m}} t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

The matrix with the cosines is independent of the initial state, so that once we know this matrix, we can work out the general solution as a function of time for any initial state. The matrix is known as the *propagator*. [Although Shankar uses the symbol  $U(t)$  to refer to the propagator, it's *not* a unitary matrix. For example, its determinant is  $\cos\left(\sqrt{\frac{k}{m}} t\right) \cos\left(\sqrt{\frac{3k}{m}} t\right) \neq 1$  for  $t \neq 0$ .]

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