COUPLED MASSES ON SPRINGS - PROPERTIES OF THE PROPAGATOR

We’ll continue our study of the system of two masses coupled by springs. The system is described by the matrix equation of motion:

\[
\dot{\mathbf{x}}(t) = \mathbf{\Omega} \mathbf{x}(t)
\]  

(1)

where

\[
\mathbf{x}(t) = x_1(t) |1\rangle + x_2(t) |2\rangle
\]

(2)

in the basis

\[
|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

(3)

\[
|2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

(4)

In this basis, \( \mathbf{\Omega} \) is the operator whose matrix form is

\[
\mathbf{\Omega} = \begin{bmatrix} -2k/m & k/m \\ k/m & -2k/m \end{bmatrix}
\]

(5)

We found that the solution could be written as

\[
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \cos \left( \frac{k}{m} t + \sqrt{\frac{3k}{m}} t \right) & \cos \left( \frac{k}{m} t - \sqrt{\frac{3k}{m}} t \right) \\ \cos \left( \frac{k}{m} t - \sqrt{\frac{3k}{m}} t \right) & \cos \left( \frac{k}{m} t + \sqrt{\frac{3k}{m}} t \right) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}
\]

(6)

In compact form, we can write this as

\[
|\mathbf{x}(t)\rangle = \mathbf{U}(t) |\mathbf{x}(0)\rangle
\]

(7)

where the propagator operator is defined as
\[ U(t) = \frac{1}{2} \begin{bmatrix} \cos \sqrt{\frac{k}{m}}t + \cos \sqrt{\frac{3k}{m}}t & \cos \sqrt{\frac{k}{m}}t - \cos \sqrt{\frac{3k}{m}}t \\ \cos \sqrt{\frac{k}{m}}t - \cos \sqrt{\frac{3k}{m}}t & \cos \sqrt{\frac{k}{m}}t + \cos \sqrt{\frac{3k}{m}}t \end{bmatrix} \] (8)

From (1), we can operate on both sides of (7) with the operator \( \frac{d^2}{dt^2} - \Omega \) to get

\[ \left( \frac{d^2}{dt^2} - \Omega \right) |x(t)\rangle = \left( \frac{d^2}{dt^2} - \Omega \right) U(t) |x(0)\rangle = 0 \] (9)

Since the initial positions \( |x(0)\rangle \) are arbitrary and contains no time dependence, the matrix \( U(t) \) satisfies the differential equation

\[ \frac{d^2 U(t)}{dt^2} = \Omega U(t) \] (10)

By direct calculation (I used Maple, but you can do it by hand using the usual rules for matrix multiplication, although it’s quite tedious), we can show that \( \Omega \) and \( U \) commute and, since both \( \Omega \) and \( U \) are hermitian, they are simultaneously diagonalizable. We already worked out the eigenvectors of \( \Omega \):

\[ |I\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \] (11)

\[ |II\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \] (12)

Since \( \Omega \) is not degenerate, these must also be the eigenvectors of \( U \), so the unitary matrix

\[ D = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \] (13)

can be used to diagonalize \( U \) according to
\[
D^\dagger UD = \frac{1}{4} \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
\cos \sqrt{\frac{k}{m}} t + \cos \sqrt{\frac{3k}{m}} t & \cos \sqrt{\frac{k}{m}} t - \cos \sqrt{\frac{3k}{m}} t \\
\cos \sqrt{\frac{k}{m}} t - \cos \sqrt{\frac{3k}{m}} t & \cos \sqrt{\frac{k}{m}} t + \cos \sqrt{\frac{3k}{m}} t
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\]

(14)

\[
= \frac{1}{2} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
\cos \sqrt{\frac{k}{m}} t & \cos \sqrt{\frac{3k}{m}} t \\
\cos \sqrt{\frac{k}{m}} t & -\cos \sqrt{\frac{3k}{m}} t
\end{bmatrix}
\]

(15)

\[
= \begin{bmatrix}
\cos \sqrt{\frac{k}{m}} t & 0 \\
0 & \cos \sqrt{\frac{3k}{m}} t
\end{bmatrix}
\]

(16)

This matches the diagonal form for \( U \) given as equation 1.8.43 in Shankar’s book. The diagonal entries are the eigenvalues of \( U(t) \).