COUPLED MASSES ON SPRINGS - PROPERTIES OF THE PROPAGATOR

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercise 1.8.12.

We'll continue our study of the system of two masses coupled by springs. The system is described by the matrix equation of motion:

$$|\ddot{x}(t)\rangle = \Omega|x(t)\rangle \tag{1}$$

where

$$|x(t)\rangle = x_1(t)|1\rangle + x_2(t)|2\rangle \tag{2}$$

in the basis

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3}$$

$$|2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4}$$

In this basis, Ω is the operator whose matrix form is

$$\Omega = \begin{bmatrix} -2\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -2\frac{k}{m} \end{bmatrix} \tag{5}$$

We found that the solution could be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \cos\sqrt{\frac{k}{m}}t + \cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{k}{m}}t - \cos\sqrt{\frac{3k}{m}}t \\ \cos\sqrt{\frac{k}{m}}t - \cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{k}{m}}t + \cos\sqrt{\frac{3k}{m}}t \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{6}$$

In compact form, we can write this as

$$|x(t)\rangle = U(t)|x(0)\rangle \tag{7}$$

where the propagator operator is defined as

$$U(t) \equiv \frac{1}{2}\left[\begin{array}{cc} \cos\sqrt{\frac{k}{m}}t+\cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{k}{m}}t-\cos\sqrt{\frac{3k}{m}}t \\ \cos\sqrt{\frac{k}{m}}t-\cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{k}{m}}t+\cos\sqrt{\frac{3k}{m}}t \end{array}\right] \tag{8}$$

From 1, we can operate on both sides of 7 with the operator $\frac{d^2}{dt^2}-\Omega$ to get

$$\left(\frac{d^2}{dt^2}-\Omega\right)|x(t)\rangle = \left(\frac{d^2}{dt^2}-\Omega\right)U(t)\,|x(0)\rangle = 0 \tag{9}$$

Since the initial positions $|x(0)\rangle$ are arbitrary and contains no time dependence, the matrix $U(t)$ satisfies the differential equation

$$\frac{d^2U(t)}{dt^2} = \Omega U(t) \tag{10}$$

By direct calculation (I used Maple, but you can do it by hand using the usual rules for matrix multiplication, although it's quite tedious), we can show that Ω and U commute and, since both Ω and U are hermitian, they are simultaneously diagonalizable. We already worked out the eigenvectors of Ω:

$$|I\rangle = \frac{1}{\sqrt{2}}\left[\begin{array}{c} 1 \\ 1 \end{array}\right] \tag{11}$$

$$|II\rangle = \frac{1}{\sqrt{2}}\left[\begin{array}{c} 1 \\ -1 \end{array}\right] \tag{12}$$

Since Ω is not degenerate, these must also be the eigenvectors of U, so the unitary matrix

$$D = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right] \tag{13}$$

can be used to diagonalize U according to

$$D^{\dagger}UD=\frac{1}{4}\left[\begin{array}{cc}1 & 1\\1 & -1\end{array}\right]\left[\begin{array}{cc}\cos\sqrt{\frac{k}{m}}t+\cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{k}{m}}t-\cos\sqrt{\frac{3k}{m}}t\\\cos\sqrt{\frac{k}{m}}t-\cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{k}{m}}t+\cos\sqrt{\frac{3k}{m}}t\end{array}\right]\left[\begin{array}{cc}1 & 1\\1 & -1\end{array}\right] \tag{14}$$

$$=\frac{1}{2}\left[\begin{array}{cc}1 & 1\\1 & -1\end{array}\right]\left[\begin{array}{cc}\cos\sqrt{\frac{k}{m}}t & \cos\sqrt{\frac{3k}{m}}t\\\cos\sqrt{\frac{k}{m}}t & -\cos\sqrt{\frac{3k}{m}}t\end{array}\right] \tag{15}$$

$$=\left[\begin{array}{cc}\cos\sqrt{\frac{k}{m}}t & 0\\0 & \cos\sqrt{\frac{3k}{m}}t\end{array}\right] \tag{16}$$

This matches the diagonal form for U given as equation 1.8.43 in Shankar's book. The diagonal entries are the eigenvalues of $U\left(t\right)$.

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