

COUPLED MASSES ON SPRINGS - PROPERTIES OF THE PROPAGATOR

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Exercise 1.8.12.

We'll continue our study of the system of two masses coupled by springs. The system is described by the matrix equation of motion:

$$(0.1) \quad |\ddot{x}(t)\rangle = \Omega |x(t)\rangle$$

where

$$(0.2) \quad |x(t)\rangle = x_1(t) |1\rangle + x_2(t) |2\rangle$$

in the basis

$$(0.3) \quad |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(0.4) \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In this basis, Ω is the operator whose matrix form is

$$(0.5) \quad \Omega = \begin{bmatrix} -2\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -2\frac{k}{m} \end{bmatrix}$$

We found that the solution could be written as

$$(0.6) \quad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \cos\sqrt{\frac{k}{m}}t + \cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{k}{m}}t - \cos\sqrt{\frac{3k}{m}}t \\ \cos\sqrt{\frac{k}{m}}t - \cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{k}{m}}t + \cos\sqrt{\frac{3k}{m}}t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

In compact form, we can write this as

$$(0.7) \quad |x(t)\rangle = U(t) |x(0)\rangle$$

where the propagator operator is defined as

$$(0.8) \quad U(t) \equiv \frac{1}{2} \begin{bmatrix} \cos \sqrt{\frac{k}{m}}t + \cos \sqrt{\frac{3k}{m}}t & \cos \sqrt{\frac{k}{m}}t - \cos \sqrt{\frac{3k}{m}}t \\ \cos \sqrt{\frac{k}{m}}t - \cos \sqrt{\frac{3k}{m}}t & \cos \sqrt{\frac{k}{m}}t + \cos \sqrt{\frac{3k}{m}}t \end{bmatrix}$$

From 0.1, we can operate on both sides of 0.7 with the operator $\frac{d^2}{dt^2} - \Omega$ to get

$$(0.9) \quad \left(\frac{d^2}{dt^2} - \Omega \right) |x(t)\rangle = \left(\frac{d^2}{dt^2} - \Omega \right) U(t) |x(0)\rangle = 0$$

Since the initial positions $|x(0)\rangle$ are arbitrary and contains no time dependence, the matrix $U(t)$ satisfies the differential equation

$$(0.10) \quad \frac{d^2 U(t)}{dt^2} = \Omega U(t)$$

By direct calculation (I used Maple, but you can do it by hand using the usual rules for matrix multiplication, although it's quite tedious), we can show that Ω and U commute and, since both Ω and U are hermitian, they are simultaneously diagonalizable. We already worked out the eigenvectors of Ω :

$$(0.11) \quad |I\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(0.12) \quad |II\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Since Ω is not degenerate, these must also be the eigenvectors of U , so the unitary matrix

$$(0.13) \quad D = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

can be used to diagonalize U according to

(0.14)

$$D^\dagger U D = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \cos \sqrt{\frac{k}{m}} t + \cos \sqrt{\frac{3k}{m}} t & \cos \sqrt{\frac{k}{m}} t - \cos \sqrt{\frac{3k}{m}} t \\ \cos \sqrt{\frac{k}{m}} t - \cos \sqrt{\frac{3k}{m}} t & \cos \sqrt{\frac{k}{m}} t + \cos \sqrt{\frac{3k}{m}} t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(0.15)

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \cos \sqrt{\frac{k}{m}} t & \cos \sqrt{\frac{3k}{m}} t \\ \cos \sqrt{\frac{k}{m}} t & -\cos \sqrt{\frac{3k}{m}} t \end{bmatrix}$$

(0.16)

$$= \begin{bmatrix} \cos \sqrt{\frac{k}{m}} t & 0 \\ 0 & \cos \sqrt{\frac{3k}{m}} t \end{bmatrix}$$

This matches the diagonal form for U given as equation 1.8.43 in Shankar's book. The diagonal entries are the eigenvalues of $U(t)$.

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