

EXPONENTIALS OF OPERATORS - BAKER-CAMPBELL-HAUSDORFF FORMULA

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 1.9.

Although the result in this post isn't covered in Shankar's book, it's a result that is frequently used in quantum theory, so it's worth including at this point.

We've seen how to define a function of an operator if that function can be expanded in a power series. A common operator function is the exponential:

$$f(\Omega) = e^{i\Omega} \quad (1)$$

If Ω is hermitian, the exponential $e^{i\Omega}$ is unitary. If we try to calculate the exponential of two operators such as e^{A+B} , the result isn't as simple as we might hope if A and B don't commute. To see the problem, we can write this out as a power series

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} \quad (2)$$

$$= I + A + B + \frac{1}{2}(A+B)(A+B) + \dots \quad (3)$$

$$= I + A + B + \frac{1}{2}(A^2 + AB + BA + B^2) + \dots \quad (4)$$

The problem appears first in the fourth term in the series, since we can't condense the $AB + BA$ sum into $2AB$ if $[A, B] \neq 0$. In fact, the expansion of $e^A e^B$ can be written entirely in terms of the commutators of A and B with each other, nested to increasingly higher levels. This formula is known as the Baker-Campbell-Hausdorff formula. Up to the fourth order commutator, the BCH formula gives

$$e^A e^B = \exp \left[A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [B, [B, A]]) - \frac{1}{24}[B, [A, [A, B]]] + \dots \right] \quad (5)$$

There is no known closed form expression for this result. However, an important special case that occurs frequently in quantum theory is the case

where $[A, B] = cI$, where c is a complex scalar and I is the usual identity matrix. Since cI commutes with all operators, all terms from the third order upwards are zero, and we have

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \quad (6)$$

We can prove this result as follows. Start with the operator function

$$G(t) \equiv e^{t(A+B)} e^{-tA} \quad (7)$$

where t is a scalar parameter (not necessarily time!).

From its definition,

$$G(0) = I \quad (8)$$

The inverse is

$$G^{-1}(t) = e^{tA} e^{-t(A+B)} \quad (9)$$

and the derivative is

$$\frac{dG(t)}{dt} = (A+B) e^{t(A+B)} e^{-tA} - e^{t(A+B)} e^{-tA} A \quad (10)$$

Note that we have to keep the $(A+B)$ factor to the left of the A factor because $[A, B] \neq 0$. Now we multiply:

$$G^{-1} \frac{dG}{dt} = e^{tA} e^{-t(A+B)} \left[(A+B) e^{t(A+B)} e^{-tA} - e^{t(A+B)} e^{-tA} A \right] \quad (11)$$

$$= e^{tA} (A+B) e^{-tA} - A \quad (12)$$

$$= e^{tA} A e^{-tA} + e^{tA} B e^{-tA} - A \quad (13)$$

$$= e^{tA} B e^{-tA} \quad (14)$$

$$= B + t [A, B] \quad (15)$$

$$= B + ctI \quad (16)$$

We used Hadamard's lemma in the penultimate line, which in this case reduces to

$$e^{tA} B e^{-tA} = B + t [A, B] \quad (17)$$

because $[A, B] = cI$ so all higher order commutators are zero.

We end up with an expression in which A has disappeared. This gives the differential equation for G :

$$G^{-1} \frac{dG}{dt} = B + ctI \quad (18)$$

We try a solution of the form (this apparently appears from divine inspiration):

$$G(t) = e^{\alpha t B} e^{\beta c t^2} \quad (19)$$

From which we get

$$G^{-1} = e^{-\alpha t B} e^{-\beta c t^2} \quad (20)$$

$$\frac{dG}{dt} = (\alpha B + 2\beta c t) e^{\alpha t B} e^{\beta c t^2} \quad (21)$$

$$G^{-1} \frac{dG}{dt} = \alpha B + 2\beta c t \quad (22)$$

Comparing this to 18, we have

$$\alpha = 1 \quad (23)$$

$$\beta = \frac{1}{2} \quad (24)$$

$$G(t) = e^{tB} e^{\frac{1}{2} c t^2} \quad (25)$$

Setting this equal to the original definition of G in 7 and then taking $t = 1$ we have

$$e^{A+B} e^{-A} = e^B e^{c/2} \quad (26)$$

$$e^{A+B} = e^B e^A e^{\frac{1}{2} c} \quad (27)$$

$$= e^B e^A e^{\frac{1}{2} [A, B]} \quad (28)$$

If we swap A with B and use the fact that $A + B = B + A$, and also $[A, B] = -[B, A]$, we have

$$e^{A+B} = e^A e^B e^{-\frac{1}{2} [A, B]} \quad (29)$$

This is the restricted form of the BCH formula for the case where $[A, B]$ is a scalar.

PINGBACKS

Pingback: Time-dependent propagators

Pingback: Time reversal, antiunitary operators and Wigner's theorem

Pingback: Linear chain of oscillators - External force, unitary operator

Pingback: Linear chain of oscillators - External force, ground state