LAGRANGIANS FOR HARMONIC OSCILLATORS

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The Euler-Lagrange equations of motion, derived from the principle of least action are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$ (1)

where $q_i$ and $\dot{q}_i$ are the generalized coordinates and velocities, respectively. Here are a couple of simple examples of how these equations can be used to derive equations of motion.

Example 1. The harmonic oscillator. We have a mass $m$ sliding on a frictionless horizontal surface with a spring of spring constant $k$ connected between one end of the mass and a fixed support. The horizontal displacement of the mass from its equilibrium position is given by $x$, with $x < 0$ when the mass moves to the left, compressing the spring, and $x > 0$ when it moves to the right, stretching the spring.

For systems where the potential energy $V(q_i)$ is independent of the velocities $\dot{q}_i$, the Lagrangian can be written as

$$L = T - V$$ (2)

where $T$ is the kinetic energy. In the case of the mass

$$T = \frac{1}{2} m \dot{x}^2$$ (3)

$$V = \frac{1}{2} k x^2$$ (4)

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$ (5)

described earlier

The equation of motion is
which is the familiar equation for the force on the mass equal to $-kx$.

**Example 2.** We can revisit the problem of two masses coupled by three springs, as described earlier. In this case, we have two coordinates $x_1$ and $x_2$. The total kinetic energy is

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$$

(8)

The total potential energy is

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}kx_2^2$$

(9)

$$= k(x_1^2 + x_2^2 - x_1x_2)$$

(10)

The Lagrangian and equations of motion are then

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - k(x_1^2 + x_2^2 - x_1x_2)$$

(11)

\[
\begin{align*}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} &= m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} &= m\ddot{x}_2 + 2kx_2 - kx_1 = 0
\end{align*}
\]

(12) (13)

This gives the same equations of motion we had earlier

$$\ddot{x}_1 = -2k\frac{x_1}{m} + \frac{k}{m}x_2$$

(14)

$$\ddot{x}_2 = \frac{k}{m}x_1 - 2\frac{k}{m}x_2$$

(15)

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