LAGRANGIAN FOR THE TWO-BODY PROBLEM

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.3; Exercise 2.3.1.

A fundamental problem in classical physics is the two-body problem, in which two masses interact via a potential $V(\mathbf{r}_1-\mathbf{r}_2)$ that depends only on the relative positions of the two masses. In such a case, the Lagrangian can be decoupled so that the problem gets reduced to a one-body problem.

The Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}=0 \tag{0.1}$$

where q_i and $\dot{q}_i$ are the generalized coordinates and velocities, respectively. For systems where the potential energy $V(q_i)$ is independent of the velocities $\dot{q}_i$, the Lagrangian can be written as

$$L=T-V \tag{0.2}$$

where T is the kinetic energy. In terms of the absolute positions and velocities, we have

$$L=\frac{1}{2}m_1\left|\dot{\mathbf{r}}_1\right|^2+\frac{1}{2}m_2\left|\dot{\mathbf{r}}_2\right|^2-V(\mathbf{r}_1-\mathbf{r}_2) \tag{0.3}$$

To decouple this equation, we define two new position vectors:

$$\begin{aligned}\mathbf{r} &\equiv \mathbf{r}_1-\mathbf{r}_2 &(0.4)\\ \mathbf{r}_{CM} &\equiv \frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2} &(0.5)\end{aligned}$$

Here $\mathbf{r}$ is the relative position, and $\mathbf{r}_{CM}$ is the position of the centre of mass.

We can invert these equations to get

$$\mathbf{r}_1 = \mathbf{r} + \mathbf{r}_2 \tag{0.6}$$

$$(m_1 + m_2)\,\mathbf{r}_{CM} = m_1\mathbf{r} + (m_1 + m_2)\,\mathbf{r}_2 \tag{0.7}$$

$$\mathbf{r}_2 = \mathbf{r}_{CM} - \frac{m_1}{m_1 + m_2}\mathbf{r} \tag{0.8}$$

$$\mathbf{r}_1 = \mathbf{r}_{CM} - \frac{m_2}{m_1 + m_2}\mathbf{r} \tag{0.9}$$

To decouple the Lagrangian, we insert these last two equations into 0.3.

$$m_1\,|\dot{\mathbf{r}}_1|^2 = m_1\left[\dot{\mathbf{r}}_{CM} - \frac{m_2}{m_1 + m_2}\dot{\mathbf{r}}\right]\cdot\left[\dot{\mathbf{r}}_{CM} - \frac{m_2}{m_1 + m_2}\dot{\mathbf{r}}\right] \tag{0.10}$$

$$= m_1\,|\dot{\mathbf{r}}_{CM}|^2 - 2\frac{m_1 m_2}{m_1 + m_2}\dot{\mathbf{r}}_{CM}\cdot\dot{\mathbf{r}} + m_1\left(\frac{m_2}{m_1 + m_2}\right)^2|\dot{\mathbf{r}}|^2 \tag{0.11}$$

$$m_2\,|\dot{\mathbf{r}}_2|^2 = m_2\left[\dot{\mathbf{r}}_{CM} + \frac{m_1}{m_1 + m_2}\dot{\mathbf{r}}\right]\cdot\left[\dot{\mathbf{r}}_{CM} + \frac{m_1}{m_1 + m_2}\dot{\mathbf{r}}\right] \tag{0.12}$$

$$= m_2\,|\dot{\mathbf{r}}_{CM}|^2 + 2\frac{m_1 m_2}{m_1 + m_2}\dot{\mathbf{r}}_{CM}\cdot\dot{\mathbf{r}} + m_2\left(\frac{m_1}{m_1 + m_2}\right)^2|\dot{\mathbf{r}}|^2 \tag{0.13}$$

(0.14)
$$\frac{1}{2}m_1\,|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2\,|\dot{\mathbf{r}}_2|^2 = \frac{1}{2}(m_1 + m_2)\,|\dot{\mathbf{r}}_{CM}|^2 + \frac{1}{2}\frac{m_1 m_2^2 + m_2 m_1^2}{(m_1 + m_2)^2}|\dot{\mathbf{r}}|^2$$

$$= \frac{1}{2}(m_1 + m_2)\,|\dot{\mathbf{r}}_{CM}|^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}|\dot{\mathbf{r}}|^2 \tag{0.15}$$

The Lagrangian 0.3 thus becomes

$$L = \frac{1}{2}(m_1 + m_2)\,|\dot{\mathbf{r}}_{CM}|^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}|\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \tag{0.16}$$

$$\equiv L_{CM} + L_r \tag{0.17}$$

with

$$L_{CM} \equiv \frac{1}{2}(m_1 + m_2)\,|\dot{\mathbf{r}}_{CM}|^2 \tag{0.18}$$

$$L_r \equiv \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}|\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \tag{0.19}$$

Thus L decouples into two Lagrangians, one of which depends only on $\dot{\mathbf{r}}_{CM}$ and the other of which depends only on $\mathbf{r}$ and $\dot{\mathbf{r}}$. The absence of $\mathbf{r}_{CM}$ means that, from 0.1

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_{i,CM}}=\frac{d}{dt}\frac{\partial L_{CM}}{\partial \dot{r}_{i,CM}}=\frac{m_1+m_2}{2}\frac{d\dot{r}_{i,CM}}{dt}=0 \tag{0.20}$$

$$\dot{r}_{i,CM}=\text{constant} \tag{0.21}$$

which is separately true for each component of $\dot{\mathbf{r}}_{CM}$, which shows that the velocity of the centre of mass is a constant, as we'd expect for an isolated two-body system with no external force.

From the other Lagrangian, we get

$$\frac{m_1 m_2}{m_1+m_2}\ddot{\mathbf{r}}=-\nabla V\left(\mathbf{r}\right) \tag{0.22}$$

which is the equation of motion of a single particle of mass $\frac{m_1 m_2}{m_1+m_2}$, called the *reduced mass*. Viewed from the centre of mass frame, where $\dot{\mathbf{r}}_{CM}=0$, $\mathbf{r}$ becomes the absolute position of the reduced mass. We can transform the result back to the 'absolute' frame by using 0.4.

PINGBACKS

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