LAGRANGIAN FOR THE TWO-BODY PROBLEM

A fundamental problem in classical physics is the two-body problem, in which two masses interact via a potential $V(r_1 - r_2)$ that depends only on the relative positions of the two masses. In such a case, the Lagrangian can be decoupled so that the problem gets reduced to a one-body problem.

The Euler-Lagrange equations are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$  \hspace{1cm} (1)

where $q_i$ and $\dot{q}_i$ are the generalized coordinates and velocities, respectively. For systems where the potential energy $V(q_i)$ is independent of the velocities $\dot{q}_i$, the Lagrangian can be written as

$$L = T - V$$  \hspace{1cm} (2)

where $T$ is the kinetic energy. In terms of the absolute positions and velocities, we have

$$L = \frac{1}{2} m_1 |\dot{r}_1|^2 + \frac{1}{2} m_2 |\dot{r}_2|^2 - V(r_1 - r_2)$$  \hspace{1cm} (3)

To decouple this equation, we define two new position vectors:

$$\mathbf{r} \equiv r_1 - r_2$$  \hspace{1cm} (4)

$$\mathbf{r}_{CM} \equiv \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$  \hspace{1cm} (5)

Here $\mathbf{r}$ is the relative position, and $\mathbf{r}_{CM}$ is the position of the centre of mass.

We can invert these equations to get
\[ \mathbf{r}_1 = \mathbf{r} + \mathbf{r}_2 \]  
(6)

\[ (m_1 + m_2) \mathbf{r}_{CM} = m_1 \mathbf{r} + (m_1 + m_2) \mathbf{r}_2 \]  
(7)

\[ \mathbf{r}_2 = \mathbf{r}_{CM} - \frac{m_1}{m_1 + m_2} \mathbf{r} \]  
(8)

\[ \mathbf{r}_1 = \mathbf{r}_{CM} - \frac{m_2}{m_1 + m_2} \mathbf{r} \]  
(9)

To decouple the Lagrangian, we insert these last two equations into 3.

\[ m_1 |\dot{\mathbf{r}}_1|^2 = m_1 \left[ \dot{\mathbf{r}}_{CM} - \frac{m_2}{m_1 + m_2} \dot{\mathbf{r}} \right] \cdot \left[ \dot{\mathbf{r}}_{CM} - \frac{m_2}{m_1 + m_2} \dot{\mathbf{r}} \right] \]  
(10)

\[ = m_1 |\dot{\mathbf{r}}_{CM}|^2 - 2 \frac{m_1 m_2}{m_1 + m_2} \dot{\mathbf{r}}_{CM} \cdot \dot{\mathbf{r}} + m_1 \left( \frac{m_2}{m_1 + m_2} \right)^2 |\dot{\mathbf{r}}|^2 \]  
(11)

\[ m_2 |\dot{\mathbf{r}}_2|^2 = m_2 \left[ \dot{\mathbf{r}}_{CM} + \frac{m_1}{m_1 + m_2} \dot{\mathbf{r}} \right] \cdot \left[ \dot{\mathbf{r}}_{CM} + \frac{m_1}{m_1 + m_2} \dot{\mathbf{r}} \right] \]  
(12)

\[ = m_2 |\dot{\mathbf{r}}_{CM}|^2 + 2 \frac{m_1 m_2}{m_1 + m_2} \dot{\mathbf{r}}_{CM} \cdot \dot{\mathbf{r}} + m_2 \left( \frac{m_1}{m_1 + m_2} \right)^2 |\dot{\mathbf{r}}|^2 \]  
(13)

\[ \frac{1}{2} m_1 |\dot{\mathbf{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\mathbf{r}}_2|^2 = \frac{1}{2} (m_1 + m_2) |\dot{\mathbf{r}}_{CM}|^2 + \frac{1}{2} m_1 m_2 + m_2 m_1 |\dot{\mathbf{r}}|^2 \]  
(14)

\[ = \frac{1}{2} (m_1 + m_2) |\dot{\mathbf{r}}_{CM}|^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\dot{\mathbf{r}}|^2 \]  
(15)

The Lagrangian thus becomes

\[ L = \frac{1}{2} (m_1 + m_2) |\dot{\mathbf{r}}_{CM}|^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \]  
(16)

\[ \equiv L_{CM} + L_r \]  
(17)

with

\[ L_{CM} \equiv \frac{1}{2} (m_1 + m_2) |\dot{\mathbf{r}}_{CM}|^2 \]  
(18)

\[ L_r \equiv \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \]  
(19)
Thus $L$ decouples into two Lagrangians, one of which depends only on $\dot{r}_{CM}$ and the other of which depends only on $\dot{r}$ and $\dot{r}$. The absence of $r_{CM}$ means that, from

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}_{i,CM}} = \frac{d}{dt} \frac{\partial L_{CM}}{\partial \dot{r}_{i,CM}} = \frac{m_1 + m_2}{2} \frac{d\dot{r}_{i,CM}}{dt} = 0 \quad (20)$$

which is separately true for each component of $\dot{r}_{CM}$, which shows that the velocity of the centre of mass is a constant, as we’d expect for an isolated two-body system with no external force.

From the other Lagrangian, we get

$$m_1 \dot{r}_{i,CM} = -\nabla V(r) \quad (21)$$

which is the equation of motion of a single particle of mass $\frac{m_1 m_2}{m_1 + m_2}$, called the reduced mass. Viewed from the centre of mass frame, where $\dot{r}_{CM} = 0$, $r$ becomes the absolute position of the reduced mass. We can transform the result back to the ‘absolute’ frame by using

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