

HAMILTONIANS FOR HARMONIC OSCILLATORS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.5; Exercises 2.5.2 - 2.5.3.

Here are a couple of examples of equations of motion using the Hamiltonian formalism. First, we look at the simple harmonic oscillator, in which we have a mass m sliding on a frictionless horizontal surface. The mass is connected to a spring with constant k , with the other end of the spring connected to a fixed support.

The Hamiltonian is given by

$$(0.1) \quad H(q, p) = \sum_i p_i \dot{q}_i - L(q, \dot{q})$$

where the velocities \dot{q}_i are expressed in terms of the positions q_i and momenta p_i . In this case, we have, using the coordinate x as the displacement from equilibrium

$$(0.2) \quad L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$(0.3) \quad p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$(0.4) \quad \dot{x} = \frac{p}{m}$$

$$(0.5) \quad L(x, \dot{x}(x, p)) = \frac{p^2}{2m} - \frac{1}{2}kx^2$$

$$(0.6) \quad H = \frac{p^2}{m} - \left(\frac{p^2}{2m} - \frac{1}{2}kx^2 \right)$$

$$(0.7) \quad = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

We can now apply Hamilton's canonical equations:

$$(0.8) \quad \frac{\partial H}{\partial p} = \dot{x}$$

$$(0.9) \quad -\frac{\partial H}{\partial x} = \dot{p}$$

We get

$$(0.10) \quad \frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}$$

$$(0.11) \quad -\frac{\partial H}{\partial x} = -kx = \dot{p}$$

We thus get a pair of first order ODEs which can be solved in the usual way, given $x(0)$ and $p(0)$. The second order ODE that we got by using the Lagrangian method can be obtained by differentiating the first equation and plugging it into the second:

$$(0.12) \quad \ddot{x} = \frac{\dot{p}}{m}$$

$$(0.13) \quad = -\frac{k}{m}x$$

From 0.7 we see that, since in the absence of external force, the total energy $H = T + V = E$ is a constant,

$$(0.14) \quad \frac{p^2}{2m} + \frac{1}{2}kx^2 = E = \text{constant}$$

This can be written as the equation of an ellipse:

$$(0.15) \quad \frac{p^2}{b^2} + \frac{x^2}{a^2} = 1$$

where

$$(0.16) \quad a^2 = \frac{2E}{k}$$

$$(0.17) \quad b^2 = 2mE$$

We can use the Hamiltonian formalism to get the equations of motion of the coupled harmonic oscillator. From our Lagrangian treatment, we had

$$(0.18) \quad L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - k(x_1^2 + x_2^2 - x_1x_2)$$

Converting to coordinates and momenta, we have

$$(0.19) \quad p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$$

$$(0.20) \quad \dot{x}_i = \frac{p_i}{m}$$

$$(0.21) \quad H = \sum_i p_i \dot{x}_i - L(x, \dot{x})$$

$$(0.22) \quad = \frac{1}{m} (p_1^2 + p_2^2) - \left[\frac{1}{2m} m (p_1^2 + p_2^2) - k (x_1^2 + x_2^2 - x_1 x_2) \right]$$

$$(0.23) \quad = \frac{1}{2m} (p_1^2 + p_2^2) + k (x_1^2 + x_2^2 - x_1 x_2)$$

Applying the canonical equations gives

$$(0.24) \quad \frac{\partial H}{\partial p_i} = \frac{p_i}{m} = \dot{x}_i$$

$$(0.25) \quad -\frac{\partial H}{\partial x_1} = -2kx_1 + kx_2 = \dot{p}_1$$

$$(0.26) \quad -\frac{\partial H}{\partial x_2} = -2kx_2 + kx_1 = \dot{p}_2$$

Again, by taking the derivative of the first line and substituting into the last two lines, we get back the previous equations of motion:

$$(0.27) \quad \ddot{x}_1 = -2\frac{k}{m}x_1 + \frac{k}{m}x_2$$

$$(0.28) \quad \ddot{x}_2 = \frac{k}{m}x_1 - 2\frac{k}{m}x_2$$