HAMILTONIAN FOR THE TWO-BODY PROBLEM

Here we derive the equations of motion of the two-body problem using the Hamiltonian formalism.

The Hamiltonian is given by

\[ H(q,p) = \sum_i p_i \dot{q}_i - L(q,\dot{q}) \] (1)

where the velocities \( \dot{q}_i \) are expressed in terms of the positions \( q_i \) and momenta \( p_i \). In this case, we start with the Lagrangian in terms of the centre of mass position \( r_{CM} \) and the relative position \( r \) of mass 2 to mass 1.

\[ L = \frac{1}{2} (m_1 + m_2) |\dot{r}_{CM}|^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\dot{r}|^2 - V(r) \] (2)

\[ = \frac{M}{2} |\dot{r}_{CM}|^2 + \mu \frac{1}{2} |\dot{r}|^2 - V(r) \] (3)

where \( M = m_1 + m_2 \) is the total mass and \( \mu = \frac{m_1 m_2}{m_1 + m_2} \) is the reduced mass.

There are potentially 6 velocity components and 6 coordinate components in the Lagrangian, but the 3 components of \( r_{CM} \) do not appear, which simplifies things a bit. To convert to a Hamiltonian, we need the momenta

\[ p_i = \frac{\partial L}{\partial \dot{q}_i} \] (4)

The \( x \) component of momentum of the centre of mass is

\[ p_{CM,x} = \frac{\partial L}{\partial \dot{r}_{CM,x}} = M \dot{r}_{CM,x} \] (5)

The other two components of the centre of mass velocity, and of the relative velocity, have a similar form, and in general we can write

\[ p_{CM,i} = M \dot{r}_{CM,i} \] (6)

\[ p_i = \mu \dot{r}_i \] (7)

In vector notation, this becomes
\[ \dot{r}_{CM} = \frac{p_{CM}}{M} \quad (8) \]
\[ \dot{\mathbf{r}} = \frac{\mathbf{p}}{\mu} \quad (9) \]
\[ |\dot{r}_{CM}|^2 = \frac{|p_{CM}|^2}{M^2} \quad (10) \]
\[ |\mathbf{r}|^2 = \frac{|\mathbf{p}|^2}{\mu^2} \quad (11) \]

The Lagrangian thus becomes
\[ L = \frac{|p_{CM}|^2}{2M} + \frac{|\mathbf{p}|^2}{2\mu} - V(\mathbf{r}) \quad (12) \]

The Hamiltonian is
\[ H = \mathbf{p} \cdot \dot{\mathbf{r}} + p_{CM} \cdot \dot{r}_{CM} - L \]
\[ = \frac{|\mathbf{p}|^2}{\mu} + \frac{|p_{CM}|^2}{M} - \left( \frac{|p_{CM}|^2}{2M} + \frac{|\mathbf{p}|^2}{2\mu} - V(\mathbf{r}) \right) \quad (13) \]
\[ = \frac{|p_{CM}|^2}{2M} + \frac{|\mathbf{p}|^2}{2\mu} + V(\mathbf{r}) \quad (14) \]

Once we’ve got the Hamiltonian, we can apply Hamilton’s canonical equations to get the equations of motion.
\[ \frac{\partial H}{\partial p_i} = \dot{r}_i \quad (15) \]
\[ -\frac{\partial H}{\partial r_i} = \dot{p}_i \quad (16) \]

Since \( r_{CM} \) does not appear in the Hamiltonian, we have
\[ \dot{p}_{CM} = 0 \quad (17) \]
\[ p_{CM} = \text{constant} \quad (18) \]

so the momentum of the centre of mass does not change, as expected.

For \( \mathbf{r} \), we have
\[ \frac{\partial H}{\partial p_i} = \frac{p_i}{\mu} = \dot{r}_i \tag{20} \]

\[ \frac{\partial H}{\partial r_i} = \frac{\partial V}{\partial r_i} = -\dot{p}_i \tag{21} \]

The first equation tells us nothing new, while the second is just Newton’s law for a central force: \( \dot{p} = -\nabla V \).

PINGBACKS

Pingback: [Canonical transformations: a few more examples](link)