CYCLIC COORDINATES AND POISSON BRACKETS

Hamilton’s canonical equations are:

\[
\begin{align*}
\frac{\partial H}{\partial p_i} &= \dot{q}_i \\
-\frac{\partial H}{\partial q_i} &= \dot{p}_i
\end{align*}
\]

(1) (2)

If a coordinate \(q_i\) is missing in the Hamiltonian (that is, \(H\) is independent of \(q_i\)), then

\[
\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0
\]

(3)

Thus the conjugate momentum \(p_i\) is conserved. Such a missing coordinate \(q_i\) is known as a cyclic coordinate. [I’m not sure of the origin of this term. Again Google doesn’t provide a definitive answer.]

There is a general method for calculating the rate of change of some function \(\omega(p,q)\) that depends on the momenta and coordinates, but not explicitly on the time (\(\omega\) is allowed to depend implicitly on time since \(p\) and/or \(q\) can depend on time). The time derivative can then be written using the chain rule:

\[
\frac{d\omega}{dt} = \sum_i \left( \frac{\partial \omega}{\partial q_i} \dot{q}_i + \frac{\partial \omega}{\partial p_i} \dot{p}_i \right)
\]

(4)

\[
= \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial H}{\partial q_i} \right)
\]

(5)

\[\equiv \{\omega, H\}\]

(6)

where in the second line we used Hamilton’s equations [1] and [2]. The last line defines the Poisson bracket of the function \(\omega\) with the Hamiltonian \(H\). We can see that if \(\{\omega, H\} = 0\), the function \(\omega\) is conserved.

Since \(\{H, H\} = 0\) automatically, the total energy (represented by the Hamiltonian) is conserved, provided there is no explicit time dependence.
Such a time dependence can arise if the system is subject to some external force, for example.

From the definition 5 we can derive a few fundamental properties of Poisson brackets. We'll consider a general Poisson bracket between two arbitrary functions \( \omega(p,q) \) and \( \lambda(p,q) \). Then

\[
\{ \omega, \lambda \} = \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right)
\] (7)

\[
= - \sum_i \left( \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} - \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} \right)
\] (8)

\[
= - \sum_i \left( \frac{\partial \lambda}{\partial q_i} \frac{\partial \omega}{\partial p_i} - \frac{\partial \lambda}{\partial p_i} \frac{\partial \omega}{\partial q_i} \right)
\] (9)

\[
= - \{ \lambda, \omega \}
\] (10)

A Poisson bracket is distributive, in the sense that

\[
\{ \omega, \lambda + \sigma \} = \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial (\lambda + \sigma)}{\partial p_i} \right) - \frac{\partial \omega}{\partial p_i} \left( \frac{\partial (\lambda + \sigma)}{\partial q_i} \right)
\] (11)

\[
= \sum_i \left( \frac{\partial \omega}{\partial q_i} \left[ \frac{\partial \lambda}{\partial p_i} + \frac{\partial \sigma}{\partial p_i} \right] - \frac{\partial \omega}{\partial p_i} \left[ \frac{\partial \lambda}{\partial q_i} + \frac{\partial \sigma}{\partial q_i} \right] \right)
\] (12)

\[
= \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) + \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \sigma}{\partial q_i} \right)
\] (13)

\[
= \{ \omega, \lambda \} + \{ \omega, \sigma \}
\] (14)

One more identity is useful, which we can derive using the product rule:

\[
\{ \omega, \lambda \sigma \} = \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial (\lambda \sigma)}{\partial p_i} \right) - \frac{\partial \omega}{\partial p_i} \left( \frac{\partial (\lambda \sigma)}{\partial q_i} \right)
\] (15)

\[
= \sum_i \sigma \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) + \sum_i \lambda \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \sigma}{\partial q_i} \right)
\] (16)

\[
= \{ \omega, \lambda \} \sigma + \{ \omega, \sigma \} \lambda
\] (17)

The Poisson brackets involving the coordinates \( q_i \) and momenta \( p_i \) turn up frequently, so it’s worth deriving them in detail. We have

\[
\{ q_i, q_j \} = \sum_k \left( \frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right) = 0
\] (18)
This follows because, in the Hamiltonian formalism, the \( q_i \)s and \( p_i \)s are independent variables, so \( \frac{\partial q_j}{\partial p_k} = \frac{\partial p_j}{\partial q_k} = 0 \) for all \( j \) and \( k \). For the same reason, we have

\[
\{p_i, p_j\} = \sum_k \left( \frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = 0 \tag{19}
\]

The mixed Poisson bracket is a different story, however:

\[
\{q_i, p_j\} = \sum_k \left( \frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = \sum_k \delta_{ik} \delta_{jk} - 0 = \delta_{ij} \tag{20}
\]

Hamilton’s equations 1 and 2 can be written using Poisson brackets by setting \( \omega \) equal to \( q_i \) and \( p_i \) respectively in 6:

\[
\dot{q}_i = \{q_i, H\} \tag{23}
\]
\[
\dot{p}_i = \{p_i, H\} \tag{24}
\]

**Example.** In two dimensions, we have a Hamiltonian:

\[
H = p_x^2 + p_y^2 + ax^2 + by^2 \tag{25}
\]

If \( a = b \), then in polar coordinates, the only coordinate appearing in \( H \) is the radial distance from the origin \( r = \sqrt{x^2 + y^2} \), which means that the polar angle \( \theta \) is a cyclic coordinate. This means that the conjugate momentum \( p_\theta \) must be conserved. That is,

\[
\dot{p}_\theta = \{p_\theta, H\} = 0 \tag{26}
\]

However, \( p_\theta \) is the angular momentum \( \ell_z \), so this just says that angular momentum is conserved.

To see this explicitly, it’s easier to convert to polar coordinates. From Hamilton’s equations
\[ \dot{x} = \frac{\partial H}{\partial p_x} = 2p_x \quad (27) \]
\[ \dot{y} = 2p_y \quad (28) \]
\[ p_x^2 + p_y^2 = \frac{1}{4}(\dot{x}^2 + \dot{y}^2) \quad (29) \]
\[ = \frac{v^2}{4} \quad (30) \]
\[ = \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) \quad (31) \]

where in the fourth line, \( v \) is the linear velocity and in the fifth line we converted this to polar coordinates. Thus the Hamiltonian becomes, in the case where \( a = b \):

\[ H = \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) + ar^2 \quad (32) \]

To find the conjugate momenta in polar coordinates, we can write out the Lagrangian. We use \( p_x\dot{x} = \frac{x^2}{2} \) and \( p_y\dot{y} = \frac{y^2}{2} \) and get

\[ L = \sum p_i \dot{q}_i - H \quad (33) \]
\[ = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) - ar^2 \quad (34) \]
\[ = \frac{1}{4}(\dot{r}^2 + r^2\dot{\theta}^2) - ar^2 \quad (35) \]

The conjugate momenta are thus

\[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}r^2\dot{\theta} \quad (36) \]
\[ p_r = \frac{\partial L}{\partial \dot{r}} = \frac{i}{2} \quad (37) \]

From this we can see that \( p_\theta \) is indeed angular momentum as it’s proportional to the product of \( r \) and the tangential velocity \( v_\theta = r\dot{\theta} \). (‘Real’ momentum and angular momentum must, of course, also contain a factor of a mass, but from the definition of the Hamiltonian above, we see that the mass has been incorporated into the momentum parameters.)

Plugging these back into (32) we get

\[ H = p_r^2 + p_\theta^2 + ar^2 \quad (38) \]

We can now calculate the Poisson brackets easily:
\[ \{ p_\theta, H \} = \sum_i \left( \frac{\partial p_\theta}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial p_\theta}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \]

\[ = 0 - \frac{\partial p_\theta}{\partial \theta} \frac{\partial H}{\partial p_\theta} = 0 \quad (40) \]

\[ \{ p_r, H \} = \sum_i \left( \frac{\partial p_r}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial p_r}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \]

\[ = 0 - \frac{\partial p_r}{\partial r} \frac{\partial H}{\partial p_r} = -2ar \quad (43) \]

Thus \( p_\theta \) (the angular momentum) is conserved, while \( p_r < 0 \), so that the object is always being pulled in towards the origin.

**Pingbacks**

- Conditions for a transformation to be canonical
- Poisson brackets are invariant under a canonical transformation
- Passive, regular and active transformations.
- The classical limit of quantum mechanics; Ehrenfest’s theorem
- Poisson brackets to commutators: classical to quantum
- Angular momentum - Poisson bracket to commutator
- Direct product of two vector spaces
- Correspondence between classical and quantum transformations
- Linear chain of oscillators - Classical treatment, Hamiltonian
- Nonrelativistic field theory - Schrödinger equation
- Poisson brackets, commutators and Jacobi identity