CONDITIONS FOR A TRANSFORMATION TO BE CANONICAL

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.7; Exercise 2.7.3.

We've seen that the Euler-Lagrange equations are invariant under canonical transformations, but in the Hamiltonian formalism where the system moves in a $2n$-dimensional phase space with n coordinates q and n momenta p, more general transformations are possible:

$$\overline{q}_i = \overline{q}_i(q,p) \tag{0.1}$$

$$\overline{p}_i = \overline{p}_i(q,p) \tag{0.2}$$

In order for such a transformation to be canonical, we require that the new variables $\overline{q}$ and $\overline{p}$ satisfy Hamilton's equations, that is

$$\frac{\partial H}{\partial \overline{p}_i} = \dot{\overline{q}}_i \tag{0.3}$$

$$-\frac{\partial H}{\partial \overline{q}_i} = \dot{\overline{p}}_i \tag{0.4}$$

In principle, then, we could check the Hamiltonian in the new coordinates to see if these equations are valid, but it would seem that whether or not a set of coordinates and momenta is canonical should be determinable from the variables themselves, and not depend on the specific Hamiltonian. Here we derive a set of conditions on the $\overline{q}$ and $\overline{p}$ that determine whether or not the transformation is canonical.

The time derivative of any function ω can be written as a Poisson bracket:

$$\dot{\omega} = \{\omega, H\} \tag{0.5}$$

For the transformed velocities, we have

$$\dot{\overline{q}}_j = \{\overline{q}_j, H\} \tag{0.6}$$

$$= \sum_i \left(\frac{\partial \overline{q}_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \overline{q}_j}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \tag{0.7}$$

Here, H is written as a function $H(q,p)$ of the original variables. If we write it as a function of the transformed variables, we can find the two derivatives of H in 0.7 by using the chain rule:

$$\frac{\partial H(\overline{q},\overline{p})}{\partial p_i} = \sum_k \left(\frac{\partial H}{\partial \overline{q}_k}\frac{\partial \overline{q}_k}{\partial p_i} + \frac{\partial H}{\partial \overline{p}_k}\frac{\partial \overline{p}_k}{\partial p_i} \right) \tag{0.8}$$

$$\frac{\partial H(\overline{q},\overline{p})}{\partial q_i} = \sum_k \left(\frac{\partial H}{\partial \overline{q}_k}\frac{\partial \overline{q}_k}{\partial q_i} + \frac{\partial H}{\partial \overline{p}_k}\frac{\partial \overline{p}_k}{\partial q_i} \right) \tag{0.9}$$

Inserting these into 0.7 we get

(0.10)

$$\dot{\overline{q}}_j = \sum_i \sum_k \left[\frac{\partial \overline{q}_j}{\partial q_i}\left(\frac{\partial H}{\partial \overline{q}_k}\frac{\partial \overline{q}_k}{\partial p_i} + \frac{\partial H}{\partial \overline{p}_k}\frac{\partial \overline{p}_k}{\partial p_i} \right) - \frac{\partial \overline{q}_j}{\partial p_i}\left(\frac{\partial H}{\partial \overline{q}_k}\frac{\partial \overline{q}_k}{\partial q_i} + \frac{\partial H}{\partial \overline{p}_k}\frac{\partial \overline{p}_k}{\partial q_i} \right) \right]$$

(0.11)

$$= \sum_k \frac{\partial H}{\partial \overline{q}_k} \sum_i \left(\frac{\partial \overline{q}_j}{\partial q_i}\frac{\partial \overline{q}_k}{\partial p_i} - \frac{\partial \overline{q}_j}{\partial p_i}\frac{\partial \overline{q}_k}{\partial q_i} \right) + \sum_k \frac{\partial H}{\partial \overline{p}_k} \sum_i \left(\frac{\partial \overline{q}_j}{\partial q_i}\frac{\partial \overline{p}_k}{\partial p_i} - \frac{\partial \overline{q}_j}{\partial p_i}\frac{\partial \overline{p}_k}{\partial q_i} \right)$$

(0.12)

$$= \sum_k \frac{\partial H}{\partial \overline{q}_k} \left\{ \overline{q}_j, \overline{q}_k \right\} + \sum_k \frac{\partial H}{\partial \overline{p}_k} \left\{ \overline{q}_j, \overline{p}_k \right\}$$

In order for this result to satisfy 0.3, we must have

$$\left\{ \overline{q}_j, \overline{q}_k \right\} = 0 \tag{0.13}$$

$$\left\{ \overline{q}_j, \overline{p}_k \right\} = \delta_{jk} \tag{0.14}$$

We can repeat the calculation for $\dot{\overline{p}}_i$:

(0.15)

$$\dot{\overline{p}}_j = \left\{\overline{p}_j, H\right\}$$

(0.16)

$$= \sum_i \left(\frac{\partial \overline{p}_j}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \overline{p}_j}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

(0.17)

$$= \sum_i \sum_k \left[\frac{\partial \overline{p}_j}{\partial q_i} \left(\frac{\partial H}{\partial \overline{q}_k} \frac{\partial \overline{q}_k}{\partial p_i} + \frac{\partial H}{\partial \overline{p}_k} \frac{\partial \overline{p}_k}{\partial p_i} \right) - \frac{\partial \overline{p}_j}{\partial p_i} \left(\frac{\partial H}{\partial \overline{q}_k} \frac{\partial \overline{q}_k}{\partial q_i} + \frac{\partial H}{\partial \overline{p}_k} \frac{\partial \overline{p}_k}{\partial q_i} \right) \right]$$

(0.18)

$$= \sum_k \frac{\partial H}{\partial \overline{q}_k} \sum_i \left(\frac{\partial \overline{p}_j}{\partial q_i} \frac{\partial \overline{q}_k}{\partial p_i} - \frac{\partial \overline{p}_j}{\partial p_i} \frac{\partial \overline{q}_k}{\partial q_i} \right) + \sum_k \frac{\partial H}{\partial \overline{p}_k} \sum_i \left(\frac{\partial \overline{p}_j}{\partial q_i} \frac{\partial \overline{p}_k}{\partial p_i} - \frac{\partial \overline{p}_j}{\partial p_i} \frac{\partial \overline{p}_k}{\partial q_i} \right)$$

(0.19)

$$= \sum_k \frac{\partial H}{\partial \overline{q}_k} \left\{\overline{p}_j, \overline{q}_k\right\} + \sum_k \frac{\partial H}{\partial \overline{p}_k} \left\{\overline{p}_j, \overline{p}_k\right\}$$

Requiring this to satsify 0.4, we have

$$\left\{\overline{p}_j, \overline{p}_k\right\} = 0 \tag{0.20}$$

$$\left\{\overline{p}_j, \overline{q}_k\right\} = -\delta_{jk} \tag{0.21}$$

The last equation is equivalent to

$$\left\{\overline{q}_j, \overline{p}_k\right\} = \delta_{jk} \tag{0.22}$$

which agrees with 0.14. Thus in order for the transformation to be canonical, the conditions are

$$\left\{\overline{q}_j, \overline{q}_k\right\} = \left\{\overline{p}_j, \overline{p}_k\right\} = 0 \tag{0.23}$$

$$\left\{\overline{q}_j, \overline{p}_k\right\} = \delta_{jk} \tag{0.24}$$

Note that these Poisson brackets require calculating the derivatives of the new variables $\overline{q}$ and $\overline{p}$ with respect to the original ones q and p, but they *don't* involve any particular Hamiltonian. Thus it's possible to determine whether or not a transformation is canonical entirely from the transformation equations 0.1 and 0.2.

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