INFINITESIMAL ROTATIONS IN CANONICAL AND NONCANONICAL TRANSFORMATIONS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 2.8; Exercises 2.8.3 - 2.8.4.

Here are a couple of examples of transformations of variables and their consequences with regard to conservation laws.

First, we look at the 2-d harmonic oscillator where the Hamiltonian is

$$H=\frac{1}{2m}\left(p_x^2+p_y^2\right)+\frac{1}{2}m\omega^2\left(x^2+y^2\right) \tag{1}$$

If we rotate the system so that both the coordinates and momenta get rotated, then

$$\bar{x} = x\cos\theta-y\sin\theta \tag{2}$$
$$\bar{y} = x\sin\theta+y\cos\theta \tag{3}$$
$$\bar{p}_x = p_x\cos\theta-p_y\sin\theta \tag{4}$$
$$\bar{p}_y = p_x\sin\theta+p_y\cos\theta \tag{5}$$

We can show by direct calculation that H is invariant under this transformation, and we can verify that this is a canonical transformation. Shankar shows in his equation 2.8.8 that the generator of this transformation is the angular momentum $\ell_z=xp_y-yp_x$.

However, if we rotate only the coordinates and not the momenta, we get the transformation:

$$\bar{x} = x\cos\theta-y\sin\theta \tag{6}$$
$$\bar{y} = x\sin\theta+y\cos\theta \tag{7}$$
$$\bar{p}_x = p_x \tag{8}$$
$$\bar{p}_y = p_y \tag{9}$$

Again, we can show by direct calculation that

$$\bar{x}^2+\bar{y}^2=x^2+y^2 \tag{10}$$

so H is also invariant under this transformation. However, this transformation is noncanonical, as we can see by calculating one of the Poisson brackets:

$$\{\bar{x},\bar{p}_x\} = \sum_i \left(\frac{\partial\bar{x}}{\partial q_i}\frac{\partial\bar{p}_x}{\partial p_i} - \frac{\partial\bar{x}}{\partial p_i}\frac{\partial\bar{p}_x}{\partial q_i}\right) \tag{11}$$

$$= \cos\theta \neq 1 \tag{12}$$

The other mixed brackets (with a coordinate and a momentum) are also not either 0 or 1 as would be required if the transformation were to be canonical.

In order for this transformation to give rise to a conservation law, we would need to find a generator g that satisfied, for an infinitesimal rotation ε:

$$\bar{q}_i = q_i + \varepsilon\frac{\partial g}{\partial p_i} \equiv q_i + \delta q_i \tag{13}$$

$$\bar{p}_i = p_i - \varepsilon\frac{\partial g}{\partial q_i} \equiv p_i + \delta p_i \tag{14}$$

For an infinitesimal rotation, the transformation 6 becomes

$$\bar{x} = x - \varepsilon y \tag{15}$$

$$\bar{y} = y + \varepsilon x \tag{16}$$

$$\bar{p}_x = p_x \tag{17}$$

$$\bar{p}_y = p_y \tag{18}$$

Therefore, the generator would have to satisfy

$$\frac{\partial g}{\partial p_x} = -y \tag{19}$$

$$\frac{\partial g}{\partial p_y} = x \tag{20}$$

$$\frac{\partial g}{\partial x} = 0 \tag{21}$$

$$\frac{\partial g}{\partial y} = 0 \tag{22}$$

The last two conditions state that g cannot depend on x or y, but integrating the first two conditions, we get

$$g = -yp_x + xp_y + f(x,y) \tag{23}$$

where f is a function that depends only on x and/or y. Thus there is no g that satisfies all four conditions, so there is no conservation law associated with a rotation of the coordinates only, even though the Hamiltonian is invariant under this transformation. Only canonical transformations that leave H invariant give rise to conservation laws.

As another example, suppose he have the one-dimensional system with

$$H = \frac{1}{2}\left(p^2 + x^2\right) \tag{24}$$

and perform a rotation in phase space, that is, in the $x - p$ plane:

$$\bar{x} = x\cos\theta - p\sin\theta \tag{25}$$
$$\bar{p} = x\sin\theta + p\cos\theta \tag{26}$$

The Hamiltonian is invariant:

$$\bar{p}^2 + \bar{x}^2 = x^2\sin^2\theta + 2xp\sin\theta\cos\theta + p^2\cos^2\theta + \tag{27}$$
$$x^2\cos^2\theta - 2xp\sin\theta\cos\theta + p^2\sin^2\theta \tag{28}$$
$$= x^2 + p^2 \tag{29}$$

The transformation is canonical as we can verify by calculating the Poisson bracket

$$\{\bar{x},\bar{p}\} = \frac{\partial\bar{x}}{\partial x}\frac{\partial\bar{p}}{\partial p} - \frac{\partial\bar{x}}{\partial p}\frac{\partial\bar{p}}{\partial x} \tag{30}$$
$$= \cos^2\theta - \left(-\sin^2\theta\right) \tag{31}$$
$$= 1 \tag{32}$$

An infinitesimal rotation gives the transformation

$$\bar{x} = x - \varepsilon p \tag{33}$$
$$\bar{p} = p + \varepsilon x \tag{34}$$

To find the generator, we need to solve 13 and 14:

$$\frac{\partial g}{\partial p} = -p \tag{35}$$
$$\frac{\partial g}{\partial x} = -x \tag{36}$$

These can be integrated to give

$$g(x,p) = -\frac{1}{2}\left(p^2 + x^2\right) + C \tag{37}$$

where C is a constant of integration. Thus the quantity that is conserved is (apart from the minus sign, which we could eliminate by rotating through $-\theta$ instead of θ) is just the original Hamiltonian, or total energy.

PINGBACKS

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