POSTULATES OF QUANTUM MECHANICS: MOMENTUM

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Sections 4.1 - 4.2; Exercises 4.2.2 - 4.2.3.

One of the postulates of quantum mechanics is that the momentum operator P in position space is given by

$$\left\langle x\left|P\right|x'\right\rangle =-i\hbar\delta'\left(x-x'\right) \tag{0.1}$$

By using the properties of the derivative of the delta function, we can find the eigenfunctions of P. We have

$$\left\langle x\left|P\right|\psi\right\rangle = \int\left\langle x\left|P\right|x'\right\rangle \left\langle x'\left|\psi\right.\right\rangle dx' \tag{0.2}$$

$$= -i\hbar\int\delta'\left(x-x'\right)\left\langle x'\left|\psi\right.\right\rangle dx' \tag{0.3}$$

$$= -i\hbar\frac{d}{dx}\left\langle x\left|\psi\right.\right\rangle \tag{0.4}$$

$$= -i\hbar\frac{d\psi\left(x\right)}{dx} \tag{0.5}$$

The eigenvector of P is $\left|p\right\rangle$ and has the property that

$$P\left|p\right\rangle =p\left|p\right\rangle \tag{0.6}$$

If we project this onto position space and use 0.5 we get

$$\left\langle x\left|P\right|\psi\right\rangle = p\left\langle x\left|p\right.\right\rangle \tag{0.7}$$

$$-i\hbar\frac{d\psi_{p}\left(x\right)}{dx} = p\psi_{p}\left(x\right) \tag{0.8}$$

where

$$\psi_{p}\left(x\right)\equiv\left\langle x\left|p\right.\right\rangle \tag{0.9}$$

Solving this differential equation and normalizing so that $\left\langle p'\left|p\right.\right\rangle =\delta\left(p-p'\right)$ we get

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \tag{0.10}$$

For an arbitrary wave function $|\psi\rangle$, if we know its position-space form, we can find its momentum-space version as follows:

$$\langle p|\psi\rangle = \int \langle p|x\rangle \langle x|\psi\rangle \, dx \tag{0.11}$$
$$= \int \psi_p^*(x) \langle x|\psi\rangle \, dx \tag{0.12}$$
$$= \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar} \psi(x) \, dx \tag{0.13}$$

This has an interesting consequence if the position-space function $\psi(x)$ is real. The probability density for finding a particle in a state with momentum p is $|\langle p|\psi\rangle|^2$, which we can write as

$$|\langle p|\psi\rangle|^2 = \langle p|\psi\rangle^* \langle p|\psi\rangle \tag{0.14}$$
$$= \frac{1}{2\pi\hbar} \int\int e^{ip(x-x')/\hbar} \psi(x) \psi(x') \, dx \, dx' \tag{0.15}$$
$$= \frac{1}{2\pi\hbar} \int\int e^{-ip(x'-x)/\hbar} \psi(x) \psi(x') \, dx \, dx' \tag{0.16}$$
$$= \frac{1}{2\pi\hbar} \int\int e^{-ip(x-x')/\hbar} \psi(x') \psi(x) \, dx \, dx' \tag{0.17}$$
$$= |\langle -p|\psi\rangle|^2 \tag{0.18}$$

In the fourth line, since x and x' are dummy integration variables, both of which are integrated over the same range, we can simply swap them without changing anything. Note that the derivation relies on $\psi(x)$ being real, since if it were complex we would have

$$|\langle p|\psi\rangle|^2 = \langle p|\psi\rangle^* \langle p|\psi\rangle \tag{0.19}$$

$$= \frac{1}{2\pi\hbar}\int\int e^{ip(x-x')/\hbar}\psi(x)\,\psi^*\left(x'\right)dx\,dx' \tag{0.20}$$

$$= \frac{1}{2\pi\hbar}\int\int e^{-ip(x'-x)/\hbar}\psi(x)\,\psi^*\left(x'\right)dx\,dx' \tag{0.21}$$

$$= \frac{1}{2\pi\hbar}\int\int e^{-ip(x-x')/\hbar}\psi\left(x'\right)\psi^*(x)\,dx\,dx' \tag{0.22}$$

$$\neq |\langle -p|\psi\rangle|^2 \tag{0.23}$$

since

$$|\langle -p|\psi\rangle|^2 = \frac{1}{2\pi\hbar}\int\int e^{-ip(x-x')/\hbar}\psi(x)\,\psi^*\left(x'\right)dx\,dx' \tag{0.24}$$

That is, for $|\langle -p|\psi\rangle|^2$ the position x' that is the argument of the $\psi^*(x')$ factor appears as the positive term ipx' in the exponential, but in 0.22 the argument of the complex conjugate wave function is x, which appears as the negative term $-ipx$ in the exponential.

Thus for any real wave function, the probability of the particle having momentum $+p$ is equal to the probability of it having $-p$, so for such wave functions, the mean momentum is always $\langle P\rangle = 0$.

As another example, suppose we have a wave function $\psi(x)$ with a mean momentum $\bar{p}$, so that

$$\langle\psi|P|\psi\rangle = \bar{p} \tag{0.25}$$

If we now multiply ψ by $e^{ip_0x/\hbar}$ where p_0 is a constant momentum, we can calculate the new mean momentum using 0.5:

$$\langle P\rangle = \left\langle e^{ip_0x/\hbar}\psi|P|e^{ip_0x/\hbar}\psi\right\rangle \tag{0.26}$$

$$= -i\hbar\int e^{-ip_0x/\hbar}\psi^*(x)\frac{d}{dx}\left(e^{ip_0x/\hbar}\psi(x)\right)dx \tag{0.27}$$

$$= -i\hbar\int e^{-ip_0x/\hbar}\psi^*\left[\frac{ip_0}{\hbar}e^{ip_0x/\hbar}\psi(x) + e^{ip_0x/\hbar}\frac{d}{dx}\psi(x)\right]dx \tag{0.28}$$

$$= \int p_0\psi^*\psi dx - i\hbar\int\psi^*(x)\frac{d}{dx}\psi(x)\,dx \tag{0.29}$$

$$= p_0 + \bar{p} \tag{0.30}$$

The first integral in the fourth line uses the fact that p_0 is constant and ψ is normalized so that

$$\int \psi^* \psi dx = 1 \tag{0.31}$$

PINGBACKS

Pingback: Free particle in the position basis
Pingback: Propagator for a Gaussian wave packet for the free particle