

POSTULATES OF QUANTUM MECHANICS: MOMENTUM

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Sections 4.1 - 4.2; Exercises 4.2.2 - 4.2.3.

One of the postulates of quantum mechanics is that the momentum operator P in position space is given by

$$\langle x|P|x'\rangle = -i\hbar\delta'(x-x') \quad (1)$$

By using the properties of the derivative of the delta function, we can find the eigenfunctions of P . We have

$$\langle x|P|\psi\rangle = \int \langle x|P|x'\rangle \langle x'|\psi\rangle dx' \quad (2)$$

$$= -i\hbar \int \delta'(x-x') \langle x'|\psi\rangle dx' \quad (3)$$

$$= -i\hbar \frac{d}{dx} \langle x|\psi\rangle \quad (4)$$

$$= -i\hbar \frac{d\psi(x)}{dx} \quad (5)$$

The eigenvector of P is $|p\rangle$ and has the property that

$$P|p\rangle = p|p\rangle \quad (6)$$

If we project this onto position space and use 5 we get

$$\langle x|P|\psi\rangle = p\langle x|p\rangle \quad (7)$$

$$-i\hbar \frac{d\psi_p(x)}{dx} = p\psi_p(x) \quad (8)$$

where

$$\psi_p(x) \equiv \langle x|p\rangle \quad (9)$$

Solving this differential equation and normalizing so that $\langle p'|p\rangle = \delta(p-p')$ we get

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \quad (10)$$

For an arbitrary wave function $|\psi\rangle$, if we know its position-space form, we can find its momentum-space version as follows:

$$\langle p|\psi\rangle = \int \langle p|x\rangle \langle x|\psi\rangle dx \quad (11)$$

$$= \int \psi_p^*(x) \langle x|\psi\rangle dx \quad (12)$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar} \psi(x) dx \quad (13)$$

This has an interesting consequence if the position-space function $\psi(x)$ is real. The probability density for finding a particle in a state with momentum p is $|\langle p|\psi\rangle|^2$, which we can write as

$$|\langle p|\psi\rangle|^2 = \langle p|\psi\rangle^* \langle p|\psi\rangle \quad (14)$$

$$= \frac{1}{2\pi\hbar} \int \int e^{ip(x-x')/\hbar} \psi(x) \psi(x') dx dx' \quad (15)$$

$$= \frac{1}{2\pi\hbar} \int \int e^{-ip(x'-x)/\hbar} \psi(x) \psi(x') dx dx' \quad (16)$$

$$= \frac{1}{2\pi\hbar} \int \int e^{-ip(x-x')/\hbar} \psi(x') \psi(x) dx dx' \quad (17)$$

$$= |\langle -p|\psi\rangle|^2 \quad (18)$$

In the fourth line, since x and x' are dummy integration variables, both of which are integrated over the same range, we can simply swap them without changing anything. Note that the derivation relies on $\psi(x)$ being real, since if it were complex we would have

$$|\langle p|\psi\rangle|^2 = \langle p|\psi\rangle^* \langle p|\psi\rangle \quad (19)$$

$$= \frac{1}{2\pi\hbar} \int \int e^{ip(x-x')/\hbar} \psi(x) \psi^*(x') dx dx' \quad (20)$$

$$= \frac{1}{2\pi\hbar} \int \int e^{-ip(x'-x)/\hbar} \psi(x) \psi^*(x') dx dx' \quad (21)$$

$$= \frac{1}{2\pi\hbar} \int \int e^{-ip(x-x')/\hbar} \psi(x') \psi^*(x) dx dx' \quad (22)$$

$$\neq |\langle -p|\psi\rangle|^2 \quad (23)$$

since

$$|\langle -p | \psi \rangle|^2 = \frac{1}{2\pi\hbar} \int \int e^{-ip(x-x')/\hbar} \psi(x) \psi^*(x') dx dx' \quad (24)$$

That is, for $|\langle -p | \psi \rangle|^2$ the position x' that is the argument of the $\psi^*(x')$ factor appears as the positive term ipx' in the exponential, but in 22 the argument of the complex conjugate wave function is x , which appears as the negative term $-ipx$ in the exponential.

Thus for any real wave function, the probability of the particle having momentum $+p$ is equal to the probability of it having $-p$, so for such wave functions, the mean momentum is always $\langle P \rangle = 0$.

As another example, suppose we have a wave function $\psi(x)$ with a mean momentum \bar{p} , so that

$$\langle \psi | P | \psi \rangle = \bar{p} \quad (25)$$

If we now multiply ψ by $e^{ip_0x/\hbar}$ where p_0 is a constant momentum, we can calculate the new mean momentum using 5:

$$\langle P \rangle = \langle e^{ip_0x/\hbar} \psi | P | e^{ip_0x/\hbar} \psi \rangle \quad (26)$$

$$= -i\hbar \int e^{-ip_0x/\hbar} \psi^*(x) \frac{d}{dx} \left(e^{ip_0x/\hbar} \psi(x) \right) dx \quad (27)$$

$$= -i\hbar \int e^{-ip_0x/\hbar} \psi^* \left[\frac{ip_0}{\hbar} e^{ip_0x/\hbar} \psi(x) + e^{ip_0x/\hbar} \frac{d}{dx} \psi(x) \right] dx \quad (28)$$

$$= \int p_0 \psi^* \psi dx - i\hbar \int \psi^*(x) \frac{d}{dx} \psi(x) dx \quad (29)$$

$$= p_0 + \bar{p} \quad (30)$$

The first integral in the fourth line uses the fact that p_0 is constant and ψ is normalized so that

$$\int \psi^* \psi dx = 1 \quad (31)$$

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