FREE PARTICLE IN THE POSITION BASIS

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 5.1, Exercise 5.1.2.

In quantum mechanics, the free particle has degenerate energy eigenstates for each energy

$$E=\frac{p^2}{2m} \tag{1}$$

where p is the momentum. The degeneracy arises because the momentum can be either positive (for a particle moving to the right) or negative (to the left):

$$p=\pm\sqrt{2mE} \tag{2}$$

Thus the most general energy eigenstate is a linear combination of the two momentum states:

$$|E\rangle=\beta\left|p=\sqrt{2mE}\right\rangle+\gamma\left|p=-\sqrt{2mE}\right\rangle \tag{3}$$

This bizarre feature of quantum mechanics means that a particle in such a state could be moving either left or right, and if we make a measurement of the momentum we force the particle into one or other of the two momentum states.

We obtained this solution by working in the momentum basis, but we can also find the solution in the position basis. In that basis, the momentum operator has the form

$$P=-i\hbar\frac{d}{dx} \tag{4}$$

The matrix elements of this operator in the position basis are

$$\left\langle x\left|P\right|x'\right\rangle=-i\hbar\delta'\left(x-x'\right) \tag{5}$$

where $\delta'(x-x')$ is the derivative of the delta function with respect to the x, not the x'. We can use the properties of this derivative to get a solution in the X basis. To be completely formal about it, the derivation of the matrix elements of P^2 in the X basis is:

$$\left\langle x\left|P^{2}\right|\psi\right\rangle = \int\int\left\langle x\left|P\right|x'\right\rangle\left\langle x'\left|P\right|x''\right\rangle\left\langle x''\left|\psi\right.\right\rangle dx'dx'' \tag{6}$$

$$= \int\int\left\langle x\left|P\right|x'\right\rangle\left(-i\hbar\delta'\left(x'-x''\right)\right)\psi\left(x''\right)dx'dx'' \tag{7}$$

$$= -i\hbar\int\left\langle x\left|P\right|x'\right\rangle\frac{d\psi\left(x'\right)}{dx'}dx' \tag{8}$$

$$= -i\hbar\int\int\left(-i\hbar\delta'\left(x-x'\right)\right)\frac{d\psi\left(x'\right)}{dx'}dx' \tag{9}$$

$$= -\hbar^{2}\frac{d^{2}}{dx^{2}}\psi\left(x\right) \tag{10}$$

In this basis, the Schrödinger equation is therefore the familiar one:

$$\frac{P^{2}}{2m}\left|\psi\right\rangle = E\left|\psi\right\rangle \tag{11}$$

$$\left\langle x\left|\frac{P^{2}}{2m}\right|\psi\right\rangle = E\psi\left(x\right) \tag{12}$$

$$-\frac{\hbar^{2}}{2m}\frac{d^{2}}{dx^{2}}\psi\left(x\right) = E\psi\left(x\right) \tag{13}$$

$$\frac{d^{2}}{dx^{2}}\psi\left(x\right) = -\frac{2mE}{\hbar^{2}}\psi\left(x\right) \tag{14}$$

This has the general solution

$$\psi\left(x\right) = \beta e^{ix\sqrt{2mE}/\hbar}+\gamma e^{-ix\sqrt{2mE}/\hbar} \tag{15}$$

[Shankar extracts a factor of $1/\sqrt{2\pi\hbar}$ but as he notes, this is arbitrary and can be absorbed into the constants β and γ as we've done here.]

In this derivation we've implicitly assumed that $E>0$, since there is no potential so a free particle can't really have a negative energy. However, if you follow through the derivation, you'll see that it works even if $E<0$. In that case, we'd get

$$\psi\left(x\right) = \beta e^{-x\sqrt{2m|E|}/\hbar}+\gamma e^{x\sqrt{2m|E|}/\hbar} \tag{16}$$

That is, the exponents in both terms are now real instead of imaginary. The problem with this is that the first term blows up for $x\to-\infty$ while the second blows up for $x\to+\infty$. Thus this function is not normalizable, even to a delta function (as was the case when $E>0$), so functions such as these when $E<0$ are not in the Hilbert space.