INFINITE SQUARE WELL - EXPANDING WELL

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References: Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 5.2, Exercise 5.2.1.

Shankar's treatment of the infinite square well is similar to that of Griffiths, which we've already covered, so we won't go through the details again. The main difference is that Shankar places the potential walls at $x=\pm\frac{L}{2}$ while Griffiths places them at $x=0$ and $x=a$. As a result, the stationary states found by Shankar are shifted to the left, with the result

$$\psi_n(x)=\begin{cases}\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L} & n=1,3,5,7,\ldots\\ \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} & n=2,4,6,\ldots\end{cases} \tag{0.1}$$

These results can be obtained from the form given by Griffiths (where we take the width of the well to be L rather than a):

$$\psi_n(x)=\sqrt{\frac{2}{L}}\sin\frac{n\pi\left(x+\frac{L}{2}\right)}{L} \tag{0.2}$$

$$=\sqrt{\frac{2}{L}}\left[\sin\frac{n\pi x}{L}\cos\frac{n\pi}{2}+\cos\frac{n\pi x}{L}\sin\frac{n\pi}{2}\right] \tag{0.3}$$

Choosing n to be even or odd gives the results in 0.1.

The specific problem we're solving here involves a particle that starts off in the ground state ($n=1$) of a square well of width L. The well then suddenly expands to a width of $2L$ symmetrically, that is, it now extends from $x=-L$ to $x=+L$. We are to find the probability that the particle will be found in the ground state of the new well.

We solved a similar problem before, but in that case the well expanded by moving its right-hand wall to the right while keeping the left-hand wall fixed, so that the particle found itself in the left half of the new, expanded well. In the present problem, the particle finds itself centred in the new expanded well. You might think that this shouldn't matter, but it turns out to make quite a difference. To calculate this probability, we need to express the original wave function in terms of the stationary states of the expanded well, which we'll refer to as $\phi_n(x)$. That is

$$\psi_1(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \tag{0.4}$$

Working with Shankar's functions 0.1 we find ϕ_n by replacing L by $2L$:

$$\phi_n(x) = \begin{cases} \frac{1}{\sqrt{L}} \cos \frac{n\pi x}{2L} & n = 1,3,5,7,\ldots \\ \frac{1}{\sqrt{L}} \sin \frac{n\pi x}{2L} & n = 2,4,6,\ldots \end{cases} \tag{0.5}$$

Using the orthonormality of the wave functions, we have

$$c_1 = \int_{-L}^{L} \psi_1(x)\,\phi_1(x)\,dx \tag{0.6}$$

$$= \int_{-L/2}^{L/2} \sqrt{\frac{2}{L}} \cos \frac{\pi x}{L} \frac{1}{\sqrt{L}} \cos \frac{\pi x}{2L} dx \tag{0.7}$$

$$= \frac{\sqrt{2}}{L} \int_{-L/2}^{L/2} \cos \frac{\pi x}{L} \cos \frac{\pi x}{2L} dx \tag{0.8}$$

$$= \frac{\sqrt{2}}{L} \int_{-L/2}^{L/2} \left(1 - 2\sin^2 \frac{\pi x}{2L}\right) \cos \frac{\pi x}{2L} dx \tag{0.9}$$

$$= \frac{8}{3\pi} \tag{0.10}$$

The limits of integration are reduced in the second line since $\psi_1(x) = 0$ if $x > \left|\frac{L}{2}\right|$.

Thus the probability of finding the particle in the new ground state is

$$|c_1|^2 = \frac{64}{9\pi^2} \tag{0.11}$$

Note that in the earlier problem where the well expanded to the right, the probability was $\frac{32}{9\pi^2}$, so the new probability is twice as much when the wave function remains centred in the new well.

We could have also done the calculation using Griffiths's well which extended from $x = 0$ to $x = L$. If this well expands symmetrically, it now runs from $x = -\frac{L}{2}$ to $x = \frac{3L}{2}$, and the stationary states of this new well are obtained by replacing $L \to 2L$ and $x \to x + \frac{L}{2}$, so we have

$$\phi_n(x) = \frac{1}{\sqrt{L}} \sin \frac{n\pi\left(x + \frac{L}{2}\right)}{2L} \tag{0.12}$$

We then get

$$c_1 = \int_{-L/2}^{3L/2} \psi_1(x)\,\phi_1(x)\,dx \tag{0.13}$$

$$= \frac{\sqrt{2}}{L}\int_0^L \sin\frac{\pi x}{L}\sin\frac{\pi\left(x+\frac{L}{2}\right)}{2L}dx \tag{0.14}$$

$$= \frac{8}{3\pi} \tag{0.15}$$

The integral can be done by expanding the second sine using the sine addition formula. (I just used Maple.)