# HARMONIC OSCILLATOR: HERMITE POLYNOMIALS AND ORTHOGONALITY OF EIGENFUNCTIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.3, Exercises 7.3.2 - 7.3.3.

The eigenfunctions of the harmonic oscillator are given by

$$\psi_n(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-m\omega x^2/2\hbar} \tag{1}$$

where $H_n(u)$ is a Hermite polynomial. The Hermite polynomials obey the recursion relation

$$H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x) \tag{2}$$

The first few Hermite polynomials are given in Shankar's equation 7.3.21, and we may use these to verify this relation for a couple of cases. Taking $n=2$ we have

$$H_3(x)=2xH_2(x)-4H_1(x) \tag{3}$$
$$=2x\left[-2\left(1-2x^2\right)\right]-4(2x) \tag{4}$$
$$=-12x+8x^3 \tag{5}$$

The last line agrees with $H_3$ as given in Shankar.

For $n=3$ we have

$$H_4(x)=2xH_3(x)-6H_2(x) \tag{6}$$
$$=2x\left[-12x+8x^3\right]-6\left[-2\left(1-2x^2\right)\right] \tag{7}$$
$$=12-48x^2+16y^4 \tag{8}$$

which again agrees with Shankar's equation.

When deriving the solution in terms of Hermite polynomials, we followed Griffiths and found that we could write the polynomials in the form

$$H_n(y)=\sum_{j=0}^{n}a_jy^j \tag{9}$$

where the coefficients $a_j$ obey the recursion relation

$$a_{j+2}=\frac{2j+1-\varepsilon}{(j+1)(j+2)}a_j \tag{10}$$

The $\varepsilon$ used by Griffiths is equivalent to $2\varepsilon$ in Shankar, so using Shankar's notation, we see that this recursion relation is the same as Shankar's equation 7.3.15:

$$C_{n+2}=C_n\frac{2n+1-2\varepsilon}{(n+1)(n+2)} \tag{11}$$

Here, we have

$$\varepsilon=\frac{E}{\hbar\omega} \tag{12}$$

where $E$ is the energy of the oscillator state.
Looking at the polynomials in Shankar's equation 7.3.21, we have

$$H_3(y)=-12\left(y-\frac{2}{3}y^3\right) \tag{13}$$

so

$$C_1 = -12 \tag{14}$$
$$C_3 = 8 \tag{15}$$

With $n=1$, we get from 11

$$C_3=-12\frac{3-2\varepsilon}{6} \tag{16}$$

However, for this state, $E=\left(3+\frac{1}{2}\right)\hbar\omega$ so $2\varepsilon=7$ and $C_3=8$ as required.
For $H_4$ we have

$$H_4(y)=12\left(1-4y^2+\frac{4}{3}y^4\right) \tag{17}$$

This means

$$C_0 = 12 \tag{18}$$
$$C_2 = -48 \tag{19}$$
$$C_4 = 16 \tag{20}$$

Here $E=\left(4+\frac{1}{2}\right)\hbar\omega$, so $2\varepsilon=9$ and

$$C_2 = 12\frac{(-8)}{2} = -48 \tag{21}$$

$$C_4 = -48\frac{5-9}{12} = 16 \tag{22}$$

We can see from the relation 2 that, given that $H_0 = 1$ and $H_1 = 2x$, all Hermite polynomials of even index contain only even powers of $x$, and all polynomials of odd index contain only odd powers of $x$. This means that all even Hermite polynomials are even functions of $x$, in the sense that $H_{2n}(-x) = H_{2n}(x)$, and all odd Hermite polynomials are odd functions of $x$, so that $H_{2n+1}(-x) = -H_{2n+1}(x)$.

If $\psi(x)$ is even and $\phi(x)$ is odd, then

$$\psi(-x)\phi(-x) = -\psi(x)\phi(x) \tag{23}$$

That is, the product $\psi(x)\phi(x)$ is an odd function. Since the integral of any odd function over an interval symmetric about $x = 0$ is zero, we have

$$\int_{-\infty}^{\infty} \psi(x)\phi(x)\,dx = 0 \tag{24}$$

Looking at the eigenfunctions 1, we see that the exponential factor is a Gaussian centred at $x = 0$ and is therefore even, so that $\psi_n$ will be even or odd depending on whether $n$ is even or odd. In particular, the integral of any even $\psi_n$ multiplied by any odd $\psi_n$ over all $x$ will be zero.

To show that pairs of even functions are also orthogonal is a bit trickier, but we can do it in the simplest case, where we consider the functions $\psi_0$ and $\psi_2$.

$$\int_{-\infty}^{\infty} \psi_0(x)\psi_2(x)\,dx = \sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{8}}\int_{-\infty}^{\infty} H_0\left(\sqrt{\frac{m\omega}{\hbar}}x\right)H_2\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-m\omega x^2/\hbar}dx \tag{25}$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{8}}\int_{-\infty}^{\infty}(1)\left[-2\left(1-2\frac{m\omega}{\hbar}x^2\right)\right]e^{-m\omega x^2/\hbar}dx \tag{26}$$

$$= -\sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{2}}\left[\sqrt{\frac{\pi\hbar}{m\omega}} - \sqrt{\frac{\pi\hbar}{m\omega}}\right] \tag{27}$$

$$= 0 \tag{28}$$

The two Gaussian integrals can be done using standard formulas as given in Shankar's Appendix A.2. (I used Maple.)

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