

HARMONIC OSCILLATOR - MEAN POSITION AND MOMENTUM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press.
Section 7.3, Exercise 7.3.5.

The energy eigenfunctions of the harmonic oscillator are

$$(0.1) \quad \psi_n(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(y) e^{-y^2/2}$$

where

$$(0.2) \quad y \equiv \sqrt{\frac{m\omega}{\hbar}} x$$

and H_n is the Hermite polynomial of order n . Since an even (odd) Hermite polynomial is an even (odd) function, ψ_n is even (odd) if n is even (odd), so we can use this fact to show that

$$(0.3) \quad \langle n|X|n\rangle = \int_{-\infty}^{\infty} x \psi_n^2(x) dx = 0$$

This follows because the square of either an even or odd function gives an even function, and x itself is odd, so the integrand is the product of an odd and even function, which is odd. The integral over any interval symmetric about $x = 0$ of an odd function is zero. Thus the mean position $\langle X \rangle$ of a particle in any of the harmonic oscillator's energy eigenstates is zero.

For the momentum P , we have

$$(0.4) \quad \langle n|P|n\rangle = -i\hbar \int_{-\infty}^{\infty} \psi_n \frac{d\psi_n}{dx} dx$$

As we showed earlier

$$(0.5) \quad -i\hbar \frac{d\psi_n}{dx} = i\sqrt{\hbar m\omega} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-y^2/2} \left[nH_{n-1} - \frac{1}{2}H_{n+1} \right]$$

If n is even (odd), then $nH_{n-1} - \frac{1}{2}H_{n+1}$ is odd (even), so the product $\psi_n \frac{d\psi_n}{dx}$ is always the product of one odd and one even function, making it odd. Thus

$$(0.6) \quad \langle n|P|n \rangle = 0$$

Thus the mean momentum $\langle P \rangle = 0$ in all energy eigenstates.

This means that the uncertainties in position and momentum are determined entirely by the mean square values:

$$(0.7) \quad (\Delta X)^2 = \langle X^2 \rangle - \langle X \rangle^2 = \langle X^2 \rangle$$

$$(0.8) \quad (\Delta P)^2 = \langle P^2 \rangle$$

We can work out these values for a couple of specific states. For $n = 1$ we have

$$(0.9) \quad \langle 1|X^2|1 \rangle = \int_{-\infty}^{\infty} x^2 \psi_1^2(x) dx$$

$$(0.10) \quad = \frac{1}{2} \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 H_1^2(y) e^{-y^2} dx$$

$$(0.11) \quad = \frac{1}{2} \frac{\hbar}{\sqrt{\pi m\omega}} \int_{-\infty}^{\infty} y^2 H_1^2(y) e^{-y^2} dy$$

$$(0.12) \quad = \frac{2\hbar}{\sqrt{\pi m\omega}} \int_{-\infty}^{\infty} y^4 e^{-y^2} dy$$

$$(0.13) \quad = \frac{2\hbar}{\sqrt{\pi m\omega}} \frac{3\sqrt{\pi}}{4}$$

$$(0.14) \quad = \frac{3\hbar}{2m\omega}$$

We've used

$$(0.15) \quad H_1(y) = 2y$$

and formula just before A.2.3 from the appendix in Shankar, which gives

$$(0.16) \quad I_4(\alpha) = \int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx$$

$$(0.17) \quad = \frac{\partial^2}{\partial \alpha^2} \int_{-\infty}^{\infty} e^{-\alpha x^2} dx$$

$$(0.18) \quad = \frac{\partial^2}{\partial \alpha^2} I_0(\alpha)$$

From formula A.2.2

$$(0.19) \quad I_4(\alpha) = \frac{\partial^2}{\partial \alpha^2} \sqrt{\frac{\pi}{\alpha}}$$

$$(0.20) \quad = \frac{3\sqrt{\pi}}{4\alpha^{5/2}}$$

Setting $\alpha = 1$ gives

$$(0.21) \quad \int_{-\infty}^{\infty} y^4 e^{-y^2} dy = \frac{3\sqrt{\pi}}{4}$$

For P , we have

$$(0.22) \quad \langle 1 | P^2 | 1 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi_1 \frac{d^2}{dx^2} \psi_1 dx$$

The derivative is

$$(0.23) \quad \frac{d^2}{dx^2} \psi_1 = \frac{d^2 \psi_1}{dy^2} \left(\frac{dy}{dx} \right)^2$$

$$(0.24) \quad = \frac{1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{m\omega}{\hbar} \frac{d^2}{dy^2} [H_1(y) e^{-y^2/2}]$$

$$(0.25) \quad = \frac{1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{m\omega}{\hbar} \frac{d^2}{dy^2} [2y e^{-y^2/2}]$$

$$(0.26) \quad = \left(\frac{m\omega}{4\pi\hbar} \right)^{1/4} \frac{m\omega}{\hbar} e^{-y^2/2} [2y^3 - 6y]$$

We can now evaluate 0.22:

(0.27)

$$-\hbar^2 \int_{-\infty}^{\infty} \psi_1 \frac{d^2}{dx^2} \psi_1 dx = -\hbar^2 \sqrt{\frac{m\omega}{4\pi\hbar}} \frac{m\omega}{\hbar} \sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{\infty} 2ye^{-y^2} 2[y^3 - 3y] dy$$

(0.28)

$$= -\frac{2m\omega\hbar}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} [y^4 - 3y^2] dy$$

(0.29)

$$= -\frac{2m\omega\hbar}{\sqrt{\pi}} \left[\frac{3\sqrt{\pi}}{4} - \frac{3\sqrt{\pi}}{2} \right]$$

(0.30)

$$= \frac{3}{2}m\omega\hbar$$

Thus for the $n = 1$ state

(0.31)

$$\Delta X = \sqrt{\frac{3\hbar}{2m\omega}}$$

(0.32)

$$\Delta P = \sqrt{\frac{3}{2}m\omega\hbar}$$

(0.33)

$$\Delta X \Delta P = \frac{3}{2}\hbar > \frac{\hbar}{2}$$

For the $n = 0$ (ground) state, we can use $H_0 = 1$ to get

(0.34)

$$\langle X^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi_0^2(x) dx$$

(0.35)

$$= \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 H_0^2(y) e^{-y^2} dx$$

(0.36)

$$= \frac{\hbar}{\sqrt{\pi m\omega}} \int_{-\infty}^{\infty} y^2 e^{-y^2} dy$$

(0.37)

$$= \frac{\hbar}{2m\omega}$$

For P :

$$(0.38) \quad \frac{d^2}{dx^2} \psi_0 = \frac{d^2 \psi_0}{dy^2} \left(\frac{dy}{dx} \right)^2$$

$$(0.39) \quad = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{m\omega}{\hbar} e^{-y^2/2} (y^2 - 1)$$

$$(0.40)$$

$$\langle P^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi_0 \frac{d^2}{dx^2} \psi_0 dx = -\hbar^2 \sqrt{\frac{m\omega}{\pi\hbar}} \frac{m\omega}{\hbar} \sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{\infty} e^{-y^2} (y^2 - 1) dy$$

$$(0.41) \quad = -\frac{m\omega\hbar}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} - \sqrt{\pi} \right]$$

$$(0.42) \quad = \frac{1}{2} m\omega\hbar$$

The uncertainty principle in this case gives

$$(0.43) \quad \Delta X \Delta P = \frac{\hbar}{2}$$

so it saturates the condition $\Delta X \Delta P \geq \frac{\hbar}{2}$.

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