HARMONIC OSCILLATOR - MIXED INITIAL STATE AND EHRENFEST'S THEOREM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.5.

We've already done an example of a harmonic oscillator in a mixed initial state, but it's useful to do this other example from Shankar so we can see how the modified Ehrenfest's theorem fits in. In this case, we start with a particle in the mixed initial state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] \tag{1}$$

The time-dependent solution is therefore

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{\sqrt{2}}\left[e^{-iE_0\hbar t}|0\rangle + e^{-iE_1 t}|1\rangle\right] & (2)\\ &= \frac{1}{\sqrt{2}}\left[e^{-i\omega t/2}|0\rangle + e^{-3i\omega t/2}|1\rangle\right] & (3)\end{aligned}$$

since the first two energies are $E_0 = \hbar\omega/2$ and $E_1 = 3\hbar\omega/2$.

The position and momentum operators can be written in terms of the raising and lowering operators

$$\begin{aligned} X &= \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right) & (4)\\ P &= i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right) & (5)\end{aligned}$$

To find the mean position and momentum, we can use these equations:

$$\langle X(0)\rangle = \langle\psi(0)|X|\psi(0)\rangle \tag{6}$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left[\langle 0| + \langle 1|\right]\left(a^\dagger + a\right)\left[|0\rangle + |1\rangle\right] \tag{7}$$

To work out the last line, remember that the stationary states are orthogonal so that $\langle 0|1\rangle = 0$, and that

$$a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{8}$$
$$a|n\rangle = \sqrt{n}\,|n-1\rangle \tag{9}$$

We therefore get

$$\langle X(0)\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}(1+1) = \sqrt{\frac{\hbar}{2m\omega}} \tag{10}$$

Doing a similar analysis for the momentum, we have

$$\langle P(0)\rangle = \langle \psi(0)|P|\psi(0)\rangle \tag{11}$$
$$= \frac{i}{2}\sqrt{\frac{\hbar m\omega}{2}}\left[\langle 0| + \langle 1|\right]\left(a^{\dagger} - a\right)\left[|0\rangle + |1\rangle\right] \tag{12}$$
$$= \sqrt{\frac{\hbar m\omega}{2}}\frac{1}{2i}\left[\langle 0| + \langle 1|\right]\left(a - a^{\dagger}\right)\left[|0\rangle + |1\rangle\right] \tag{13}$$
$$= \sqrt{\frac{\hbar m\omega}{2}}\frac{1}{2i}(1-1) \tag{14}$$
$$= 0 \tag{15}$$

We can expand these equations to give the averages of position and momentum at all times by plugging in 3:

$$\langle X(t)\rangle = \langle \psi(t)|X|\psi(t)\rangle \tag{16}$$
$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left[\langle 0|e^{i\omega t/2} + \langle 1|e^{3i\omega t/2}\right]\left(a^{\dagger} + a\right)\left[e^{-i\omega t/2}|0\rangle + e^{-3i\omega t/2}|1\rangle\right] \tag{17}$$
$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{-i\omega t} + e^{i\omega t}\right) \tag{18}$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t \tag{19}$$

$$\langle P(t)\rangle = \langle\psi(t)|P|\psi(t)\rangle \tag{20}$$

$$= \frac{i}{2}\sqrt{\frac{\hbar m\omega}{2}}\left[\langle 0|e^{i\omega t/2}+\langle 1|e^{3i\omega t/2}\right]\left(a^{\dagger}-a\right)\left[e^{-i\omega t/2}|0\rangle+e^{-3i\omega t/2}|1\rangle\right] \tag{21}$$

$$= \sqrt{\frac{\hbar m\omega}{2}}\frac{1}{2i}\left[\langle 0|e^{i\omega t/2}+\langle 1|e^{3i\omega t/2}\right]\left(a-a^{\dagger}\right)\left[e^{-i\omega t/2}|0\rangle+e^{-3i\omega t/2}|1\rangle\right] \tag{22}$$

$$= \sqrt{\frac{\hbar m\omega}{2}}\frac{1}{2i}\left(e^{-i\omega t}-e^{i\omega t}\right) \tag{23}$$

$$= -\sqrt{\frac{\hbar m\omega}{2}}\sin\omega t \tag{24}$$

Although we can calculate $\left\langle\dot{X}(t)\right\rangle$ and $\left\langle\dot{P}(t)\right\rangle$ directly by taking the time derivative, we can also do it by using Ehrenfest's theorem in the form

$$\frac{d\langle\Omega\rangle}{dt} = -\frac{i}{\hbar}\langle[\Omega,H]\rangle \tag{25}$$

for some operator Ω.

Since the energy of the oscillator in state $|n\rangle$ is $\left(n+\frac{1}{2}\right)\hbar\omega$, we can write the hamiltonian as

$$H = \hbar\omega\left(a^{\dagger}a+\frac{1}{2}\right) \tag{26}$$

We also have the commutator

$$\left[a,a^{\dagger}\right] = 1 \tag{27}$$

To use this for X and P we need the commutators $[a,H]$ and $\left[a^{\dagger},H\right]$, which amounts to finding

$$\begin{aligned}
\left[a,a^{\dagger}a\right] &= aa^{\dagger}a-a^{\dagger}aa && (28)\\
&= \left(1+a^{\dagger}a\right)a-a^{\dagger}aa && (29)\\
&= a && (30)\\
\left[a^{\dagger},a^{\dagger}a\right] &= a^{\dagger}a^{\dagger}a-a^{\dagger}aa^{\dagger} && (31)\\
&= a^{\dagger}a^{\dagger}a-a^{\dagger}\left(1+a^{\dagger}a^{\dagger}a\right) && (32)\\
&= -a^{\dagger} && (33)
\end{aligned}$$

Therefore we have

$$\begin{aligned} [a,H] &= \hbar\omega a && (34)\\ \left[a^{\dagger},H\right] &= -\hbar\omega a^{\dagger} && (35)\end{aligned}$$

Finally we get

$$\begin{aligned} \left\langle \dot{X}(t)\right\rangle &= -\frac{i}{\hbar}\left\langle [X,H]\right\rangle && (36)\\ &= -\frac{i}{\hbar}\left\langle \left[a+a^{\dagger},H\right]\right\rangle && (37)\\ &= -\frac{i}{\hbar}\hbar\omega\sqrt{\frac{\hbar}{2m\omega}}\left\langle a-a^{\dagger}\right\rangle && (38)\\ &= i\omega\sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{2}{\hbar m\omega}}\frac{1}{i}\left\langle P(t)\right\rangle && (39)\\ &= -\omega\sqrt{\frac{\hbar}{2m\omega}}\sin\omega t && (40)\end{aligned}$$

where we used 5 in the fourth line and 24 in the last line. The last line is indeed the time derivative of 19, so fortunately Ehrenfest's theorem gives the correct answer.

For the momentum, we have

$$\begin{aligned} \left\langle \dot{P}(t)\right\rangle &= -\frac{i}{\hbar}\left\langle [P,H]\right\rangle && (41)\\ &= -\frac{i}{\hbar}\left\langle \left[a^{\dagger}-a,H\right]\right\rangle && (42)\\ &= -\frac{i}{\hbar}\hbar\omega\sqrt{\frac{\hbar m\omega}{2}}i\left\langle -a^{\dagger}-a\right\rangle && (43)\\ &= \omega\sqrt{\frac{\hbar m\omega}{2}}\sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{2m\omega}{\hbar}}\left\langle -X(t)\right\rangle && (44)\\ &= -\omega\sqrt{\frac{\hbar m\omega}{2}}\cos\omega t && (45)\end{aligned}$$

which is the correct derivative of 24.

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