POISSON BRACKETS TO COMMUTATORS: CLASSICAL TO QUANTUM

Section 7.4, Exercise 7.4.7.

The postulates of quantum mechanics that we described earlier included specifications for the matrix elements of position $X$ and momentum $P$ in position space:

$$\langle x | X | x' \rangle = x \delta (x - x')$$  \hspace{1cm} (1)

$$\langle x | P | x' \rangle = -i \hbar \delta' (x - x')$$  \hspace{1cm} (2)

A more fundamental form of this postulate is to specify the commutation relation between $X$ and $P$, which is independent of the basis and is

$$[X, P] = i \hbar$$  \hspace{1cm} (3)

This allows the construction of explicit forms of the operators in other bases, such as the momentum basis, where

$$X = i \hbar \frac{d}{dp}$$  \hspace{1cm} (4)

$$P = p$$  \hspace{1cm} (5)

We can verify this by calculating the commutator by applying it to a function $f(p)$:

$$[X, P] f = i \hbar \frac{d}{dp} (p f (p)) - i \hbar p \frac{d}{dp} f (p)$$  \hspace{1cm} (6)

$$= i \hbar f (p) + i \hbar p \frac{d}{dp} f (p) - i \hbar p \frac{d}{dp} f (p)$$  \hspace{1cm} (7)

$$= i \hbar f (p)$$  \hspace{1cm} (8)

Thus (3) is satisfied in the momentum basis as well.

The standard recipe for converting a classical system to a quantum one is to first calculate the Poisson bracket for two physical quantities in the classical system, which gives
{ω, λ} = \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) \quad (9)

where q_i and p_i are the canonical coordinates and momenta. To convert to a quantum commutator, we replace the classical quantities by their quantum operator equivalents and the Poisson bracket by $i\hbar$ times the corresponding commutator. That is

$$[Ω, Λ] = i\hbar \{ω, λ\} \quad (10)$$

For the case of $X$ and $P$, we have, in classical mechanics in one dimension

$$\{x, p\} = \frac{\partial x}{\partial x} \frac{\partial p}{\partial p} - \frac{\partial x}{\partial p} \frac{\partial p}{\partial x} = 1 \quad (11)$$

so the quantum commutator is given by 3.

For other quantities, we can use the theorems on the Poisson brackets to reduce them:

$$\{ω, λ\} = -\{λ, ω\} \quad (12)$$
$$\{ω, λ + σ\} = \{ω, λ\} + \{ω, σ\} \quad (13)$$
$$\{ω, λσ\} = \{ω, λ\} σ + \{ω, σ\} λ \quad (14)$$

Quantum commutators obey similar rules

$$[Ω, Λ] = -[Λ, Ω] \quad (15)$$
$$[Ω, Λ + Γ] = [Ω, Λ] + [Ω, Γ] \quad (16)$$
$$[ΩΛ, Γ] = Ω[Λ, Γ] + [Ω, Γ] Λ \quad (17)$$

The main difference between Poisson brackets and commutators is that, for the latter, the order of the operators in the last equation can make a difference. That is, in 14 we could also have written

$$\{ω, λσ\} = σ \{ω, λ\} + λ \{ω, σ\} \quad (18)$$

since all three quantities are numerical (not operators), so multiplication commutes. In 17 it is not true in general that, for example

$$Ω[Λ, Γ] + [Ω, Γ] Λ = [Λ, Γ] Ω + [Ω, Γ] Λ \quad (19)$$

The conversion from classical to quantum mechanics can then be achieved in general by replacing

$$\{ω(x, p), λ(x, p)\} = γ(x, p) \quad (20)$$
by

\[ [\Omega(X, P), \Lambda(X, P)] = \hat{i}\hbar \Gamma(X, P) \quad (21) \]

where each of the operators in the last equation is obtained by replacing \( x \) in the first equation by \( X \) and \( p \) by \( P \). We do need to be careful with the ordering of the operators in the quantum version, however.

As an example, suppose we have

\[ \Omega = X \]
\[ \Lambda = X^2 + P^2 \quad (23) \]

In the classical version, we calculate the Poisson bracket

\[ \{\omega, \lambda\} = \{x, x^2 + p^2\} \]
\[ = \{x, x^2\} + \{x, p^2\} \]
\[ = 0 + 2\{x, p\}p \]
\[ = 2p \quad (27) \]

Thus, by our rule above, the quantum version should be

\[ [\Omega, \Lambda] = 2i\hbar P \quad (28) \]

We can verify this using 17

\[ [X, X^2 + P^2] = [X, X^2] + [X, P^2] \]
\[ = 0 - [P^2, X] \]
\[ = -P[P, X] - [P, X]P \]
\[ = -P(-\hat{i}\hbar) - (-\hat{i}\hbar)P \]
\[ = 2\hat{i}\hbar P \quad (33) \]

In this case, there is no ordering ambiguity in the quantum version, since \([X, P] = i\hbar\) is just a number.

For a second example, suppose we have

\[ \Omega = X^2 \]
\[ \Lambda = P^2 \quad (35) \]

The classical version gives us, using the relations 14, 11 and 27
\[
\{x^2, p^2\} = -\{p^2, x^2\} = -2\{p^2, x\} x = 2\{x, p^2\} x = 4px
\] (36)
(37)
(38)
(39)

In the classical case, this result is the same as \(4xp\), but because \(X\) and \(P\) don’t commute in the quantum form, we need to be careful about the ordering.

We can do the calculation:

\[
[X^2, P^2] = X [X, P^2] + [X, P^2] X
\] (40)

From (33) we have

\[
[X, P^2] = 2i\hbar P
\] (41)

so we get

\[
[X^2, P^2] = 2i\hbar (XP + PX)
\] (42)

Thus if the Poisson bracket involves a product of \(p\) and \(x\), this should be replaced by

\[
xp \text{ or } px \rightarrow \frac{1}{2} (XP + PX)
\] (43)

in the quantum version.

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