POISSON BRACKETS TO COMMUTATORS: CLASSICAL TO QUANTUM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Section 7.4, Exercise 7.4.7.

The postulates of quantum mechanics that we described earlierincluded specifications for the matrix elements of position X and momentum P in position space:

$$\left\langle x\left|X\right|x'\right\rangle = x\delta\left(x-x'\right) \tag{0.1}$$

$$\left\langle x\left|P\right|x'\right\rangle = -i\hbar\delta'\left(x-x'\right) \tag{0.2}$$

A more fundamental form of this postulate is to specify the commutation relation between X and P, which is independent of the basis and is

$$[X,P]=i\hbar \tag{0.3}$$

This allows the construction of explicit forms of the operators in other bases, such as the momentum basis, where

$$X = i\hbar\frac{d}{dp} \tag{0.4}$$

$$P = p \tag{0.5}$$

We can verify this by calculating the commutator by applying it to a function $f(p)$:

$$[X,P]f = i\hbar\frac{d}{dp}\left(pf(p)\right)-i\hbar p\frac{d}{dp}f(p) \tag{0.6}$$

$$= i\hbar f(p)+i\hbar p\frac{d}{dp}f(p)-i\hbar p\frac{d}{dp}f(p) \tag{0.7}$$

$$= i\hbar f(p) \tag{0.8}$$

Thus 0.3 is satisfied in the momentum basis as well.

The standard recipe for converting a classical system to a quantum one is to first calculate the Poisson bracketfor two physical quantities in the classical system, which gives

$$\{\omega,\lambda\}=\sum_i\left(\frac{\partial\omega}{\partial q_i}\frac{\partial\lambda}{\partial p_i}-\frac{\partial\omega}{\partial p_i}\frac{\partial\lambda}{\partial q_i}\right) \tag{0.9}$$

where q_i and p_i are the canonical coordinates and momenta. To convert to a quantum commutator, we replace the classical quantities by their quantum operator equivalents and the Poisson bracket by $i\hbar$ times the corresponding commutator. That is

$$[\Omega,\Lambda]=i\hbar\{\omega,\lambda\} \tag{0.10}$$

For the case of X and P, we have, in classical mechanics in one dimension

$$\{x,p\}=\frac{\partial x}{\partial x}\frac{\partial p}{\partial p}-\frac{\partial x}{\partial p}\frac{\partial p}{\partial x}=1 \tag{0.11}$$

so the quantum commutator is given by 0.3.

For other quantities, we can use the theorems on the Poisson bracketsto reduce them:

$$\{\omega,\lambda\} = -\{\lambda,\omega\} \tag{0.12}$$
$$\{\omega,\lambda+\sigma\} = \{\omega,\lambda\}+\{\omega,\sigma\} \tag{0.13}$$
$$\{\omega,\lambda\sigma\} = \{\omega,\lambda\}\sigma+\{\omega,\sigma\}\lambda \tag{0.14}$$

Quantum commutators obey similar rules

$$[\Omega,\Lambda] = -[\Lambda,\Omega] \tag{0.15}$$
$$[\Omega,\Lambda+\Gamma] = [\Omega,\Lambda]+[\Omega,\Gamma] \tag{0.16}$$
$$[\Omega\Lambda,\Gamma] = \Omega[\Lambda,\Gamma]+[\Omega,\Gamma]\Lambda \tag{0.17}$$

The main difference between Poisson brackets and commutators is that, for the latter, the order of the operators in the last equation can make a difference. That is, in 0.14 we could also have written

$$\{\omega,\lambda\sigma\}=\sigma\{\omega,\lambda\}+\lambda\{\omega,\sigma\} \tag{0.18}$$

since all three quantities are numerical (not operators), so multiplication commutes. In 0.17 it is *not* true in general that, for example

$$\Omega[\Lambda,\Gamma]+[\Omega,\Gamma]\Lambda=[\Lambda,\Gamma]\Omega+[\Omega,\Gamma]\Lambda \tag{0.19}$$

The conversion from classical to quantum mechanics can then be achieved in general by replacing

$$\{\omega(x,p),\lambda(x,p)\} = \gamma(x,p) \tag{0.20}$$

by

$$[\Omega(X,P),\Lambda(X,P)] = i\hbar\Gamma(X,P) \tag{0.21}$$

where each of the operators in the last equation is obtained by replacing x in the first equation by X and p by P. We do need to be careful with the ordering of the operators in the quantum version, however.

As an example, suppose we have

$$\Omega = X \tag{0.22}$$
$$\Lambda = X^2+P^2 \tag{0.23}$$

In the classical version, we calculate the Poisson bracket

$$\{\omega,\lambda\} = \left\{x,x^2+p^2\right\} \tag{0.24}$$
$$= \left\{x,x^2\right\}+\left\{x,p^2\right\} \tag{0.25}$$
$$= 0+2\{x,p\}p \tag{0.26}$$
$$= 2p \tag{0.27}$$

Thus, by our rule above, the quantum version should be

$$[\Omega,\Lambda] = 2i\hbar P \tag{0.28}$$

We can verify this using 0.17

$$\left[X,X^2+P^2\right] = \left[X,X^2\right]+\left[X,P^2\right] \tag{0.29}$$
$$= 0-\left[P^2,X\right] \tag{0.30}$$
$$= -P[P,X]-[P,X]P \tag{0.31}$$
$$= -P(-i\hbar)-(-i\hbar)P \tag{0.32}$$
$$= 2i\hbar P \tag{0.33}$$

In this case, there is no ordering ambiguity in the quantum version, since $[X,P] = i\hbar$ is just a number.

For a second example, suppose we have

$$\Omega = X^2 \tag{0.34}$$
$$\Lambda = P^2 \tag{0.35}$$

The classical version gives us, using the relations 0.14, 0.11 and 0.27

$$\left\{x^2,p^2\right\} = -\left\{p^2,x^2\right\} \tag{0.36}$$
$$= -2\left\{p^2,x\right\}x \tag{0.37}$$
$$= 2\left\{x,p^2\right\}x \tag{0.38}$$
$$= 4px \tag{0.39}$$

In the classical case, this result is the same as $4xp$, but because X and P don't commute in the quantum form, we need to be careful about the ordering.

We can do the calculation:

$$\left[X^2,P^2\right] = X\left[X,P^2\right] + \left[X,P^2\right]X \tag{0.40}$$

From 0.33 we have

$$\left[X,P^2\right] = 2i\hbar P \tag{0.41}$$

so we get

$$\left[X^2,P^2\right] = 2i\hbar\left(XP+PX\right) \tag{0.42}$$

Thus if the Poisson bracket involves a product of p and x, this should be replaced by

$$xp \text{ or } px \to \frac{1}{2}\left(XP+PX\right) \tag{0.43}$$

in the quantum version.

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