CHANGING THE POSITION BASIS WITH A UNITARY TRANSFORMATION

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The standard representation of the position and momentum operators in the position basis is

\[ X \rightarrow x \]  
\[ P \rightarrow -i\hbar \frac{d}{dx} \]  

It turns out it’s possible to modify this definition by adding some arbitrary function of position \( f(x) \) to \( P \) so we have

\[ X' \rightarrow x \]  
\[ P' \rightarrow -i\hbar \frac{d}{dx} + f(x) \]

Since any function of \( x \) commutes with \( X \), the commutation relations remain unchanged, so we have

\[ [X', P'] = i\hbar \]  

Another way of interpreting this change in operators is by using the unitary transformation of the \( X \) basis, in the form

\[ |x\rangle \rightarrow |\tilde{x}\rangle = e^{ig(x)/\hbar} |x\rangle = e^{ig(x)/\hbar} |x\rangle \]

where

\[ g(x) \equiv \int_{x'}^{x} f(x') \, dx' \]

The last equality in 6 comes from the fact that operating on \( |x\rangle \) with any function of the \( X \) operator (provided the function can be expanded in a power series) results in multiplying \( |x\rangle \) by the same function, but with the operator \( X \) replaced by the numeric position value.

To verify this works, we can calculate the matrix elements of the old \( X \) and \( P \) operators in the new basis. We have
\[ \langle \tilde{x} | X | \tilde{x}' \rangle = \left| \langle x | \exp \left( -ig(x)/\hbar \right) X \exp \left( ig(x')/\hbar \right) | x' \rangle \right|^2 \] (8)

At this stage, since the two exponentials are numerical functions and not operators, we can take them outside the bracket to

\[ \langle \tilde{x} | X | \tilde{x}' \rangle = \exp \left( -ig(x)/\hbar \right) \exp \left( ig(x')/\hbar \right) \langle x | X | x' \rangle \] (9)

\[ = x \delta(x - x') \] (10)

The exponentials cancel in the last line since the delta function is non-zero only when \( x = x' \).

The above result can also be obtained by inserting a couple of identity operators into Eq. 8:

\[ \langle \tilde{x} | X | \tilde{x}' \rangle = \exp \left( -ig(x)/\hbar \right) \exp \left( ig(x')/\hbar \right) \langle x | X | x' \rangle \] (12)

\[ = \exp \left( -ig(x)/\hbar \right) \langle x | z \rangle \langle z | x' \rangle dz \] (13)

\[ = \exp \left( -ig(x)/\hbar \right) \delta(x - z) \delta(z - x') \] (14)

\[ = \delta(x - x') \] (15)

The momentum operator works as follows. Using the original definition on the modified basis we have

\[ \langle x | e^{-ig(x)/\hbar} X e^{ig(x')/\hbar} | x' \rangle = \int \int \langle x | e^{-ig(x)/\hbar} | y \rangle \langle y | z \rangle \langle z | e^{ig(x')/\hbar} | x' \rangle dy dz \] (16)

\[ = \int \int \langle x | e^{-ig(x)/\hbar} | z \rangle \langle z | e^{ig(x')/\hbar} | x' \rangle dz \] (17)

\[ = \delta(x - x') \] (18)
\[ \langle \hat{x} | P | \hat{x}' \rangle = -i\hbar \left< x \left| e^{-ig(x)/\hbar} \frac{d}{dx'} e^{ig(x')/\hbar} \right| x' \right> = -i\hbar \left< x \left| e^{-ig(x)/\hbar} \frac{i}{\hbar} e^{ig(x')/\hbar} \frac{d}{dx'} \right| x' \right> - i\hbar \left< x \left| e^{-ig(x)/\hbar} e^{ig(x')/\hbar} \frac{d}{dx'} \right| x' \right> \] (19)

\[ = -i\hbar \left< x \left| e^{-ig(x)/\hbar} \frac{i}{\hbar} e^{ig(x')/\hbar} \frac{d}{dx'} \right| x' \right> - i\hbar \left< x \left| e^{-ig(x)/\hbar} e^{ig(x')/\hbar} \frac{d}{dx'} \right| x' \right> \] (20)

From 7 we have
\[ \frac{dg(x)}{dx} = \frac{d}{dx} \int^{x} f(x') \, dx' = f(x) \] (22)

This gives
\[ \langle \hat{x} | P | \hat{x}' \rangle = \left< x \left| e^{ig(x') - g(x)/\hbar} \left[ f(x') - i\hbar \frac{d}{dx'} \right] \right| x' \right> \] (23)

\[ = e^{ig(x') - g(x)/\hbar} \left[ f(x') - i\hbar \frac{d}{dx'} \right] \left< x | x' \right> \] (24)

\[ = e^{ig(x') - g(x)/\hbar} \left[ f(x') - i\hbar \frac{d}{dx'} \right] \delta(x - x') \] (25)

\[ = \left[ f(x) - i\hbar \frac{d}{dx} \right] \delta(x - x') \] (26)

This shows that by a unitary change of X basis [6], we transform the position and momentum operators (well, just the momentum operator, really) according to 3. We’ve multiplied the original \( |x \rangle \) states by a phase factor which depends on some function \( f(x) \). This doesn’t change the matrix elements of \( X \), but it does add \( f(x) \) to the matrix elements of \( P \). The commonly used definition of \( P \) is thus with \( f(x) = 0 \).