

HARMONIC OSCILLATOR: MOMENTUM SPACE FUNCTIONS AND HERMITE POLYNOMIAL RECURSION RELATIONS FROM RAISING AND LOWERING OPERATORS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press.
Section 7.5, Exercises 7.5.1 - 7.5.3.

Earlier, we found the position space energy eigenfunctions of the harmonic oscillator to be

$$\psi_n(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(y) e^{-y^2/2} \quad (1)$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-m\omega x^2/2\hbar} \quad (2)$$

where y in the first equation is shorthand for

$$y = \sqrt{\frac{m\omega}{\hbar}}x \quad (3)$$

It turns out that an alternative method for deriving these functions uses the lowering operator a . Shankar gives the derivation of $\psi_n(x)$ in his section 7.5, but we can use the same technique to derive the momentum space functions. We start with the ground state and use

$$a|0\rangle = 0 \quad (4)$$

In terms of X and P , we have

$$a = \sqrt{\frac{m\omega}{2\hbar}}X + i\frac{1}{\sqrt{2m\omega\hbar}}P \quad (5)$$

To find the momentum space functions, we need to express X and P in terms of p :

$$X = i\hbar\frac{d}{dp} \quad (6)$$

$$P = p \quad (7)$$

We thus have

$$\left[i\hbar\sqrt{\frac{m\omega}{2\hbar}}\frac{d}{dp} + i\frac{1}{\sqrt{2m\omega\hbar}}p \right] \psi_0(p) = 0 \quad (8)$$

If we define the auxiliary variable

$$z \equiv \frac{p}{\sqrt{\hbar m\omega}} \quad (9)$$

we get

$$\left(\frac{d}{dz} + z \right) \psi_0(z) = 0 \quad (10)$$

This has the solution

$$\psi_0(z) = Ae^{-z^2/2} \quad (11)$$

for some normalization constant A . Thus in terms of p we have

$$\psi_0(p) = Ae^{-p^2/2\hbar m\omega} \quad (12)$$

Normalizing in the usual way, making use of the Gaussian integral, we have

$$\int_{-\infty}^{\infty} \psi_0^2(p) dp = A^2 \int_{-\infty}^{\infty} e^{-p^2/\hbar m\omega} dp = 1 \quad (13)$$

$$A = \frac{1}{(\pi\hbar m\omega)^{1/4}} \quad (14)$$

This agrees with the earlier result which was obtained by solving a second-order differential equation.

We can also use a and a^\dagger to verify a couple of recursion relations for Hermite polynomials. Reverting back to position space we have

$$X = x \quad (15)$$

$$P = -i\hbar\frac{d}{dx} \quad (16)$$

so 5 becomes

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + \frac{\hbar}{\sqrt{2m\omega\hbar}}\frac{d}{dx} \quad (17)$$

Also from 5 we have, since X and P are both hermitian operators

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\frac{1}{\sqrt{2m\omega\hbar}}P \quad (18)$$

$$= \sqrt{\frac{m\omega}{2\hbar}}x - \frac{\hbar}{\sqrt{2m\omega\hbar}}\frac{d}{dx} \quad (19)$$

Defining

$$y \equiv \sqrt{\frac{m\omega}{\hbar}}x \quad (20)$$

we have

$$a = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right) \quad (21)$$

$$a^\dagger = \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right) \quad (22)$$

We also recall the normalization conditions on the raising and lowering operators:

$$a|n\rangle = \sqrt{n}|n-1\rangle \quad (23)$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad (24)$$

Applying 23 to 1 we have, after cancelling common factors from each side:

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2^n n!}}\left(y + \frac{d}{dy}\right)\left[H_n(y)e^{-y^2/2}\right] = \frac{\sqrt{n}}{\sqrt{2^{n-1}(n-1)!}}H_{n-1}(y)e^{-y^2/2} \quad (25)$$

$$\frac{1}{2\sqrt{n}}\frac{1}{\sqrt{2^{n-1}(n-1)!}}e^{-y^2/2}\left[yH_n(y) - yH_n(y) + \frac{dH_n}{dy}\right] = \frac{\sqrt{n}}{\sqrt{2^{n-1}(n-1)!}}H_{n-1}(y)e^{-y^2/2} \quad (26)$$

$$yH_n(y) - yH_n(y) + \frac{dH_n}{dy} = 2nH_{n-1}(y) \quad (27)$$

$$H'_n(y) = 2nH_{n-1}(y) \quad (28)$$

Another recursion relation for Hermite polynomials can be found as follows. We start with 22 to get

$$a + a^\dagger = \sqrt{2}y \quad (29)$$

We now apply 23 and 24 to 1. We can cancel common factors, including $e^{-y^2/2}$, from both sides to get

$$(a + a^\dagger) \psi_n = \sqrt{2}y \psi_n \quad (30)$$

$$\frac{\sqrt{2}y}{\sqrt{2^n n!}} H_n(y) = \frac{\sqrt{n}}{\sqrt{2^{n-1} (n-1)!}} H_{n-1}(y) + \frac{\sqrt{n+1}}{\sqrt{2^{n+1} (n+1)!}} H_{n+1}(y) \quad (31)$$

$$\frac{y}{\sqrt{2^{n-1} n (n-1)!}} H_n(y) = \frac{\sqrt{n}}{\sqrt{2^{n-1} (n-1)!}} H_{n-1}(y) + \frac{1}{2\sqrt{2^{n-1} n (n-1)!}} H_{n+1}(y) \quad (32)$$

$$yH_n(y) = nH_{n-1}(y) + \frac{1}{2}H_{n+1}(y) \quad (33)$$

$$H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y) \quad (34)$$