HARMONIC OSCILLATOR: MOMENTUM SPACE FUNCTIONS
AND HERMITE POLYNOMIAL RECURSION RELATIONS
FROM RAISING AND LOWERING OPERATORS

Earlier, we found the position space energy eigenfunctions of the harmonic oscillator to be

\[
\psi_n(y) = \left(\frac{m\omega}{\pi \hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(y) e^{-y^2/2}
\]

\[
\psi_n(x) = \left(\frac{m\omega}{\pi \hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right) e^{-m\omega x^2/2\hbar}
\]

where \(y\) in the first equation is shorthand for \(y = \sqrt{\frac{m\omega}{\hbar}} x\) \(\tag{3}\)

It turns out that an alternative method for deriving these functions uses the lowering operator \(a\). Shankar gives the derivation of \(\psi_n(x)\) in his section 7.5, but we can use the same technique to derive the momentum space functions. We start with the ground state and use \(a |0\rangle = 0\) \(\tag{4}\)

In terms of \(X\) and \(P\), we have

\[
a = \sqrt{\frac{m\omega}{2\hbar}} X + i \frac{1}{\sqrt{2m\omega\hbar}} P
\]

To find the momentum space functions, we need to express \(X\) and \(P\) in terms of \(p\):

\[
X = i\hbar \frac{d}{dp}
\]

\[
P = p
\]

We thus have
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\[
\left[ i\hbar \sqrt{\frac{m\omega}{2\hbar}} \frac{d}{dp} + i \frac{1}{\sqrt{2m\omega\hbar}} p \right] \psi_0(p) = 0 \tag{8}
\]

If we define the auxiliary variable

\[ z \equiv \frac{p}{\sqrt{\hbar m\omega}} \tag{9} \]

we get

\[
\left( \frac{d}{dz} + z \right) \psi_0(z) = 0 \tag{10}
\]

This has the solution

\[ \psi_0(z) = Ae^{-z^2/2} \tag{11} \]

for some normalization constant \( A \). Thus in terms of \( p \) we have

\[ \psi_0(p) = Ae^{-p^2/2\hbar m\omega} \tag{12} \]

Normalizing in the usual way, making use of the Gaussian integral, we have

\[
\int_{-\infty}^{\infty} \psi_0^2(p) \, dp = A^2 \int_{-\infty}^{\infty} e^{-p^2/\hbar m\omega} \, dp = 1 \tag{13}
\]

\[ A = \frac{1}{(\pi\hbar m\omega)^{1/4}} \tag{14} \]

This agrees with the earlier result which was obtained by solving a second-order differential equation.

We can also use \( a \) and \( a^\dagger \) to verify a couple of recursion relations for Hermite polynomials. Reverting back to position space we have

\[ X = x \tag{15} \]

\[ P = -i\hbar \frac{d}{dx} \tag{16} \]

so \[ a \] becomes

\[ a = \sqrt{\frac{m\omega}{2\hbar}} x + \frac{\hbar}{\sqrt{2m\omega\hbar}} \frac{d}{dx} \tag{17} \]

Also from \[ a \] we have, since \( X \) and \( P \) are both hermitian operators
\[ a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} X - i \frac{1}{\sqrt{2m\omega\hbar}} P \]  
\[ = \sqrt{\frac{m\omega}{2\hbar}} x - \frac{\hbar}{\sqrt{2m\omega\hbar}} \frac{d}{dx} \]  
(18)

Defining
\[ y \equiv \sqrt{\frac{m\omega}{\hbar}} x \]  
(20)

we have
\[ a = \frac{1}{\sqrt{2}} \left( y + \frac{d}{dy} \right) \]  
(21)
\[ a^\dagger = \frac{1}{\sqrt{2}} \left( y - \frac{d}{dy} \right) \]  
(22)

We also recall the normalization conditions on the raising and lowering operators:
\[ a \left| n \right> = \sqrt{n} \left| n - 1 \right> \]  
(23)
\[ a^\dagger \left| n \right> = \sqrt{n+1} \left| n + 1 \right> \]  
(24)

Applying (23) to (1) we have, after cancelling common factors from each side:
\[ \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2n_n!}} \left( y + \frac{d}{dy} \right) \left[ H_n(y)e^{-y^2/2} \right] = \frac{\sqrt{n}}{\sqrt{2^{n-1}(n-1)!}} H_{n-1}(y)e^{-y^2/2} \]  
\[ \frac{1}{2\sqrt{n}} \frac{1}{\sqrt{2^{n-1}(n-1)!}} e^{-y^2/2} \left[ yH_n(y) - yH_n(y) + \frac{dH_n}{dy} \right] = \frac{\sqrt{n}}{\sqrt{2^{n-1}(n-1)!}} H_{n-1}(y)e^{-y^2/2} \]  
(25)

\[ yH_n(y) - yH_n(y) + \frac{dH_n}{dy} = 2nH_{n-1}(y) \]  
(27)
\[ H'_n(y) = 2nH_{n-1}(y) \]  
(28)

Another recursion relation for Hermite polynomials can be found as follows. We start with (22) to get
\[ a + a^\dagger = \sqrt{2}y \quad (29) \]

We now apply \[23\] and \[24\] to \[1\]. We can cancel common factors, including \(e^{-y^2/2}\), from both sides to get

\[ (a + a^\dagger) \psi_n = \sqrt{2}y \psi_n \quad (30) \]

\[ \frac{\sqrt{2}y}{\sqrt{2^n n!}} H_n(y) = \frac{\sqrt{n}}{\sqrt{2^{n-1} (n-1)!}} H_{n-1}(y) + \frac{\sqrt{n+1}}{\sqrt{2^{n+1} (n+1)!}} H_{n+1}(y) \quad (31) \]

\[ \frac{y}{\sqrt{2^{n-1} n (n-1)!}} H_n(y) = \frac{\sqrt{n}}{\sqrt{2^{n-1} (n-1)!}} H_{n-1}(y) + \frac{1}{2 \sqrt{2^{n-1} n (n-1)!}} H_{n+1}(y) \quad (32) \]

\[ y H_n(y) = n H_{n-1}(y) + \frac{1}{2} H_{n+1}(y) \quad (33) \]

\[ H_{n+1}(y) = 2y H_n(y) - 2n H_{n-1}(y) \quad (34) \]