

PATH INTEGRALS FOR SPECIAL POTENTIALS; USE OF CLASSICAL ACTION

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 8. Section 8.6, Exercises 8.6.1 - 8.6.2.

We've seen that if we use the path integral formulation for a free particle, we get the exact propagator by considering only one path (the classical path) between the starting point (x', t') and the end point (x, t) . In this case, the propagator has the form

$$(0.1) \quad U(x, t; x', t') = A(t) e^{iS_{cl}/\hbar}$$

where S_{cl} is the classical action. It turns out that this form is true for a wider set of potentials, beyond just the free particle. The general form of the potential for which this is true is

$$(0.2) \quad V = a + bx + cx^2 + d\dot{x} + e\dot{x}\dot{x}$$

where a, b, c, d and e are constants. The general expression for the propagator is (where we're taking the starting time to be $t' = 0$):

$$(0.3) \quad U(x, t; x') = \int_{x'}^x e^{iS[x(t'')]/\hbar} \mathfrak{D}[x(t'')]$$

where the notation $\mathfrak{D}[x(t'')]$ means an integration over all possible paths from x' to x in the given time interval.

For a given path, we can write the location of the particle $x(t'')$ as composed of its position on the classical path $x_{cl}(t'')$ plus the deviation $y(t'')$ from the classical path:

$$(0.4) \quad x(t'') = x_{cl}(t'') + y(t'')$$

As the endpoints are fixed

$$(0.5) \quad y(0) = y(t) = 0$$

Also, since for any given potential and choice of endpoints, $x_{cl}(t'')$ is fixed for all times, it is effectively a constant with regard to the path integration. Therefore

$$(0.6) \quad dx = dy$$

Making these substitutions into 0.3, we get, using Shankar's slightly misleading notation:

$$(0.7) \quad U(x, t; x') = \int_0^0 e^{iS[x_{cl}(t'') + y(t'')]/\hbar} \mathfrak{D}[y(t'')]$$

Usually, when the limits on an integral are the same, the integral evaluates to zero. However, in this case, the notation $\int_0^0 \mathfrak{D}[y(t'')]$ means that y starts and ends at zero, but covers all possible paths between these endpoints.

The action is the integral of the Lagrangian which, for the potential 0.2 is

$$(0.8) \quad L = T - V$$

$$(0.9) \quad = \frac{1}{2}m\dot{x}^2 - a - bx - cx^2 - d\dot{x} - ex\dot{x}$$

Because L is quadratic in both x and \dot{x} , we can expand it in a Taylor series up to second order without any approximation. That is

$$(0.10) \quad L(x_{cl} + y, \dot{x}_{cl} + \dot{y}) = L(x_{cl}, \dot{x}_{cl}) + \left. \frac{\partial L}{\partial x} \right|_{x_{cl}} y + \left. \frac{\partial L}{\partial \dot{x}} \right|_{x_{cl}} \dot{y} +$$

$$(0.11) \quad \frac{1}{2} \left(\left. \frac{\partial^2 L}{\partial x^2} \right|_{x_{cl}} y^2 + 2 \left. \frac{\partial^2 L}{\partial x \partial \dot{x}} \right|_{x_{cl}} y\dot{y} + \left. \frac{\partial^2 L}{\partial \dot{x}^2} \right|_{x_{cl}} \dot{y}^2 \right)$$

Look first at the last two terms on the RHS of the first line. Using the equations of motion, we have

$$(0.12) \quad \left. \frac{\partial L}{\partial x} \right|_{x_{cl}} = \frac{d}{dt} \left(\left. \frac{\partial L}{\partial \dot{x}} \right|_{x_{cl}} \right)$$

To get the action, we need to integrate the Lagrangian over the time interval of interest. Integrating these two terms gives

(0.13)

$$\int_0^t \left[\frac{\partial L}{\partial x} \Big|_{x_{cl}} y + \frac{\partial L}{\partial \dot{x}} \Big|_{x_{cl}} \dot{y} \right] dt'' = \int_0^t \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \Big|_{x_{cl}} \right) y + \frac{\partial L}{\partial x} \Big|_{x_{cl}} \dot{y} \right] dt''$$

$$(0.14) \quad = \left(\frac{\partial L}{\partial \dot{x}} \Big|_{x_{cl}} \right) y \Big|_0^t - \int_0^t \frac{\partial L}{\partial \dot{x}} \Big|_{x_{cl}} \dot{y} dt'' + \int_0^t \frac{\partial L}{\partial x} \Big|_{x_{cl}} \dot{y} dt''$$

$$(0.15) \quad = 0$$

where we integrated the first term by parts. The integrated term in the second line is zero because $y = 0$ at both endpoints, and the last two terms cancel each other.

Returning to 0.11, we can calculate the three second derivatives explicitly:

$$(0.16) \quad \frac{1}{2} \frac{\partial^2 L}{\partial x^2} = -c$$

$$(0.17) \quad \frac{\partial^2 L}{\partial x \partial \dot{x}} \Big|_{x_{cl}} = -e$$

$$(0.18) \quad \frac{1}{2} \frac{\partial^2 L}{\partial \dot{x}^2} = m$$

The integral of the first term on the RHS of 0.10 is just the classical action, so we get for the propagator 0.7:

(0.19)

$$U(x, t; x') = e^{iS_{cl}/\hbar} \int_0^0 \exp \left[\frac{i}{\hbar} \int_0^t \left(\frac{m\dot{y}^2}{2} - cy^2 - ey\dot{y} \right) dt'' \right] \mathfrak{D}[y(t'')]$$

The remaining path integral can still be difficult to evaluate, but we can observe a few properties that it has. First, for any given path in the path integral, we must be able to express both y and \dot{y} as functions of time t'' , so the complete path integral can depend only on the end time t (and, of course, on the constants m , c and e). That is, the propagator will always have the form 0.1:

$$(0.20) \quad U(x, t; x', t') = A(t) e^{iS_{cl}/\hbar}$$

We have already evaluated the integral for the free particle where $c = e = 0$ and we found there that

$$(0.21) \quad U(x, t; x') = \sqrt{\frac{m}{2\pi\hbar it}} e^{iS_{cl}/\hbar}$$

Since the constant b doesn't appear in 0.19, the propagator must have the same form for the more general case where $V = a + bx$. For more complex potentials, such as the harmonic oscillator, the function $A(t)$ will in general have a different form and will have to be calculated explicitly in these cases.

As an example, we'll consider the case of a particle subject to a constant force in the x direction, so that the potential is given by

$$(0.22) \quad V(x) = -fx$$

This gives a constant force of

$$(0.23) \quad F = -\frac{dV}{dx} = f$$

and thus a constant acceleration of f/m . For such a particle, its classical position is (from first year physics)

$$(0.24) \quad x_{cl}(t'') = x_0 + v_0 t'' + \frac{1}{2} \frac{f}{m} t''^2$$

$$(0.25) \quad \dot{x}_{cl}(t'') = v_0 + \frac{f}{m} t''$$

To find x_0 and v_0 , we impose boundary conditions. At $t'' = 0$

$$(0.26) \quad x_{cl}(0) = x_0 = x'$$

At $t'' = t$, its position is

$$(0.27) \quad x_{cl}(t) = x = x' + v_0 t + \frac{f}{2m} t^2$$

This gives

$$(0.28) \quad v_0 = \frac{x - x'}{t} - \frac{f}{2m} t$$

The classical Lagrangian is

(0.29)

$$L = T - V$$

(0.30)

$$= \frac{1}{2} m \dot{x}_{cl}^2 + f x_{cl}$$

(0.31)

$$= \frac{1}{2} m \left(v_0 + \frac{f}{m} t'' \right)^2 + f \left(x_0 + v_0 t'' + \frac{1}{2} \frac{f}{m} t''^2 \right)$$

(0.32)

$$= \frac{1}{2} m \left(\frac{x - x'}{t} - \frac{f}{2m} t + \frac{f}{m} t'' \right)^2 + f \left(x' + \left(\frac{x - x'}{t} - \frac{f}{2m} t \right) t'' + \frac{1}{2} \frac{f}{m} t''^2 \right)$$

Note that t is a constant, as it is the time of the endpoint of the motion. To find the classical action, we must integrate this from $t'' = 0$ to t . The integral is a straightforward integral of a quadratic in t'' , although the algebra is tedious if done by hand, so is best done with Maple.

(0.33)

$$S_{cl} = \int_0^t L dt''$$

$$(0.34) \quad = \frac{1}{3} \frac{f^2 t^3}{m} + \left(\frac{x - x'}{t} - \frac{1}{2} \frac{ft}{m} \right) ft^2 + \frac{1}{2} m \left(\frac{x - x'}{t} - \frac{1}{2} \frac{ft}{m} \right)^2 t + fxt$$

$$(0.35) \quad = -\frac{f^2 t^3}{24m} + \frac{1}{2} ft(x + x') + \frac{m(x - x')^2}{2t}$$

From 0.21, this gives a propagator of

(0.36)

$$U(x, t; x') = \sqrt{\frac{m}{2\pi\hbar it}} \exp \left[\frac{i}{\hbar} \left(-\frac{f^2 t^3}{24m} + \frac{1}{2} ft(x + x') + \frac{m(x - x')^2}{2t} \right) \right]$$

This agrees with Shankar's result in his equation 5.4.31.

As another example, consider the harmonic oscillator, where the potential is

$$(0.37) \quad V = \frac{1}{2} m \omega^2 x^2$$

This potential is also of the form 0.2, so the propagator must have the form 0.20. This time, however, since $c \neq 0$, the function $A(t)$ will probably not have the form used in 0.21. The best we can say therefore is that

$$(0.38) \quad U(x, t; x', t') = A(t) e^{iS_{cl}/\hbar}$$

where $A(t)$ has the form (from 0.19):

$$(0.39) \quad A(t) = \int_0^0 \exp \left[\frac{i}{\hbar} \int_0^t \left(\frac{m\dot{y}^2}{2} - \frac{1}{2} m\omega^2 y^2 \right) dt'' \right] \mathfrak{D}[y(t'')]$$

We worked out the classical action for the harmonic oscillator earlier and found

$$(0.40) \quad S_{cl} = \frac{m\omega}{2 \sin \omega t} [(x'^2 + x^2) \cos \omega t - 2x'x]$$

where the particle is at x' at $t'' = 0$ and at x at $t'' = t$. The propagator is therefore

$$(0.41) \quad U(x, t; x') = A(t) \exp \left[\frac{im\omega}{2\hbar \sin \omega t} ((x'^2 + x^2) \cos \omega t - 2x'x) \right]$$

with $A(t)$ given by 0.39.

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