

PATH INTEGRALS FOR SPECIAL POTENTIALS; USE OF CLASSICAL ACTION

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 8. Section 8.6, Exercises 8.6.1 - 8.6.2.

We've seen that if we use the path integral formulation for a free particle, we get the exact propagator by considering only one path (the classical path) between the starting point (x', t') and the end point (x, t) . In this case, the propagator has the form

$$U(x, t; x', t') = A(t) e^{iS_{cl}/\hbar} \quad (1)$$

where S_{cl} is the classical action. It turns out that this form is true for a wider set of potentials, beyond just the free particle. The general form of the potential for which this is true is

$$V = a + bx + cx^2 + d\dot{x} + e\dot{x}^2 \quad (2)$$

where a, b, c, d and e are constants. The general expression for the propagator is (where we're taking the starting time to be $t' = 0$):

$$U(x, t; x') = \int_{x'}^x e^{iS[x(t'')]/\hbar} \mathfrak{D}[x(t'')] \quad (3)$$

where the notation $\mathfrak{D}[x(t'')]$ means an integration over all possible paths from x' to x in the given time interval.

For a given path, we can write the location of the particle $x(t'')$ as composed of its position on the classical path $x_{cl}(t'')$ plus the deviation $y(t'')$ from the classical path:

$$x(t'') = x_{cl}(t'') + y(t'') \quad (4)$$

As the endpoints are fixed

$$y(0) = y(t) = 0 \quad (5)$$

Also, since for any given potential and choice of endpoints, $x_{cl}(t'')$ is fixed for all times, it is effectively a constant with regard to the path integration. Therefore

$$dx = dy \quad (6)$$

Making these substitutions into 3, we get, using Shankar's slightly misleading notation:

$$U(x, t; x') = \int_0^0 e^{iS[x_{cl}(t'') + y(t'')]/\hbar} \mathfrak{D}[y(t'')] \quad (7)$$

Usually, when the limits on an integral are the same, the integral evaluates to zero. However, in this case, the notation $\int_0^0 \mathfrak{D}[y(t'')]$ means that y starts and ends at zero, but covers all possible paths between these endpoints.

The action is the integral of the Lagrangian which, for the potential 2 is

$$L = T - V \quad (8)$$

$$= \frac{1}{2}m\dot{x}^2 - a - bx - cx^2 - d\dot{x} - ex\dot{x} \quad (9)$$

Because L is quadratic in both x and \dot{x} , we can expand it in a Taylor series up to second order without any approximation. That is

$$L(x_{cl} + y, \dot{x}_{cl} + \dot{y}) = L(x_{cl}, \dot{x}_{cl}) + \left. \frac{\partial L}{\partial x} \right|_{x_{cl}} y + \left. \frac{\partial L}{\partial \dot{x}} \right|_{x_{cl}} \dot{y} + \quad (10)$$

$$\frac{1}{2} \left(\left. \frac{\partial^2 L}{\partial x^2} \right|_{x_{cl}} y^2 + 2 \left. \frac{\partial^2 L}{\partial x \partial \dot{x}} \right|_{x_{cl}} y\dot{y} + \left. \frac{\partial^2 L}{\partial \dot{x}^2} \right|_{x_{cl}} \dot{y}^2 \right) \quad (11)$$

Look first at the last two terms on the RHS of the first line. Using the equations of motion, we have

$$\left. \frac{\partial L}{\partial x} \right|_{x_{cl}} = \frac{d}{dt} \left(\left. \frac{\partial L}{\partial \dot{x}} \right|_{x_{cl}} \right) \quad (12)$$

To get the action, we need to integrate the Lagrangian over the time interval of interest. Integrating these two terms gives

$$\int_0^t \left[\left. \frac{\partial L}{\partial x} \right|_{x_{cl}} y + \left. \frac{\partial L}{\partial \dot{x}} \right|_{x_{cl}} \dot{y} \right] dt'' = \int_0^t \left[\frac{d}{dt} \left(\left. \frac{\partial L}{\partial \dot{x}} \right|_{x_{cl}} \right) y + \left. \frac{\partial L}{\partial \dot{x}} \right|_{x_{cl}} \dot{y} \right] dt'' \quad (13)$$

$$= \left(\left. \frac{\partial L}{\partial \dot{x}} \right|_{x_{cl}} \right) y \Big|_0^t - \int_0^t \left. \frac{\partial L}{\partial \dot{x}} \right|_{x_{cl}} \dot{y} dt'' + \int_0^t \left. \frac{\partial L}{\partial \dot{x}} \right|_{x_{cl}} \dot{y} dt'' \quad (14)$$

$$= 0 \quad (15)$$

where we integrated the first term by parts. The integrated term in the second line is zero because $y = 0$ at both endpoints, and the last two terms cancel each other.

Returning to 11, we can calculate the three second derivatives explicitly:

$$\frac{1}{2} \frac{\partial^2 L}{\partial x^2} = -c \quad (16)$$

$$\left. \frac{\partial^2 L}{\partial x \partial \dot{x}} \right|_{x_{cl}} = -e \quad (17)$$

$$\frac{1}{2} \frac{\partial^2 L}{\partial \dot{x}^2} = \frac{m}{2} \quad (18)$$

[Note that Shankar's equation 8.6.10 is wrong - the RHS should be $\frac{m}{2}$. However, his equation 8.6.11 appears to be correct. Thanks to commenter Alex for pointing this out.]

The integral of the first term on the RHS of 10 is just the classical action, so we get for the propagator 7:

$$U(x, t; x') = e^{iS_{cl}/\hbar} \int_0^0 \exp \left[\frac{i}{\hbar} \int_0^t \left(\frac{m\dot{y}^2}{2} - cy^2 - ey\dot{y} \right) dt'' \right] \mathfrak{D}[y(t'')] \quad (19)$$

The remaining path integral can still be difficult to evaluate, but we can observe a few properties that it has. First, for any given path in the path integral, we must be able to express both y and \dot{y} as functions of time t'' , so the complete path integral can depend only on the end time t (and, of course, on the constants m , c and e). That is, the propagator will always have the form 1:

$$U(x, t; x', t') = A(t) e^{iS_{cl}/\hbar} \quad (20)$$

We have already evaluated the integral for the free particle where $c = e = 0$ and we found there that

$$U(x, t; x') = \sqrt{\frac{m}{2\pi\hbar it}} e^{iS_{cl}/\hbar} \quad (21)$$

Since the constant b doesn't appear in 19, the propagator must have the same form for the more general case where $V = a + bx$. For more complex potentials, such as the harmonic oscillator, the function $A(t)$ will in general have a different form and will have to be calculated explicitly in these cases.

As an example, we'll consider the case of a particle subject to a constant force in the x direction, so that the potential is given by

$$V(x) = -fx \quad (22)$$

This gives a constant force of

$$F = -\frac{dV}{dx} = f \quad (23)$$

and thus a constant acceleration of f/m . For such a particle, its classical position is (from first year physics)

$$x_{cl}(t'') = x_0 + v_0 t'' + \frac{1}{2} \frac{f}{m} t''^2 \quad (24)$$

$$\dot{x}_{cl}(t'') = v_0 + \frac{f}{m} t'' \quad (25)$$

To find x_0 and v_0 , we impose boundary conditions. At $t'' = 0$

$$x_{cl}(0) = x_0 = x' \quad (26)$$

At $t'' = t$, its position is

$$x_{cl}(t) = x = x' + v_0 t + \frac{f}{2m} t^2 \quad (27)$$

This gives

$$v_0 = \frac{x - x'}{t} - \frac{f}{2m} t \quad (28)$$

The classical Lagrangian is

$$L = T - V \quad (29)$$

$$= \frac{1}{2} m \dot{x}_{cl}^2 + f x_{cl} \quad (30)$$

$$= \frac{1}{2} m \left(v_0 + \frac{f}{m} t'' \right)^2 + f \left(x_0 + v_0 t'' + \frac{1}{2} \frac{f}{m} t''^2 \right) \quad (31)$$

$$= \frac{1}{2} m \left(\frac{x - x'}{t} - \frac{f}{2m} t + \frac{f}{m} t'' \right)^2 + f \left(x' + \left(\frac{x - x'}{t} - \frac{f}{2m} t \right) t'' + \frac{1}{2} \frac{f}{m} t''^2 \right) \quad (32)$$

Note that t is a constant, as it is the time of the endpoint of the motion. To find the classical action, we must integrate this from $t'' = 0$ to t . The integral is a straightforward integral of a quadratic in t'' , although the algebra is tedious if done by hand, so is best done with Maple.

$$S_{cl} = \int_0^t L dt'' \quad (33)$$

$$= \frac{1}{3} \frac{f^2 t^3}{m} + \left(\frac{x-x'}{t} - \frac{1}{2} \frac{ft}{m} \right) ft^2 + \frac{1}{2} m \left(\frac{x-x'}{t} - \frac{1}{2} \frac{ft}{m} \right)^2 t + fxt \quad (34)$$

$$= -\frac{f^2 t^3}{24m} + \frac{1}{2} ft(x+x') + \frac{m(x-x')^2}{2t} \quad (35)$$

From 21, this gives a propagator of

$$U(x, t; x') = \sqrt{\frac{m}{2\pi\hbar it}} \exp \left[\frac{i}{\hbar} \left(-\frac{f^2 t^3}{24m} + \frac{1}{2} ft(x+x') + \frac{m(x-x')^2}{2t} \right) \right] \quad (36)$$

This agrees with Shankar's result in his equation 5.4.31.

As another example, consider the harmonic oscillator, where the potential is

$$V = \frac{1}{2} m\omega^2 x^2 \quad (37)$$

This potential is also of the form 2, so the propagator must have the form 20. This time, however, since $c \neq 0$, the function $A(t)$ will probably not have the form used in 21. The best we can say therefore is that

$$U(x, t; x', t') = A(t) e^{iS_{cl}/\hbar} \quad (38)$$

where $A(t)$ has the form (from 19):

$$A(t) = \int_0^0 \exp \left[\frac{i}{\hbar} \int_0^t \left(\frac{m\dot{y}^2}{2} - \frac{1}{2} m\omega^2 y^2 \right) dt'' \right] \mathfrak{D}[y(t'')] \quad (39)$$

We worked out the classical action for the harmonic oscillator earlier and found

$$S_{cl} = \frac{m\omega}{2 \sin \omega t} [(x'^2 + x^2) \cos \omega t - 2x'x] \quad (40)$$

where the particle is at x' at $t'' = 0$ and at x at $t'' = t$. The propagator is therefore

$$U(x, t; x') = A(t) \exp \left[\frac{im\omega}{2\hbar \sin \omega t} ((x'^2 + x^2) \cos \omega t - 2x'x) \right] \quad (41)$$

with $A(t)$ given by 39.

COMMENTS

Name: Alex

Error in equation 0.18 I believe the RHS should be equal to $m/2$, not m .
I think this is also incorrect in Shankar.

Time: October 25, 2017 at 11:23 pm

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Fixed now. Thanks.

PINGBACKS

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