IDENTICAL PARTICLES - BOSONS AND FERMIONS REVISITED

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 10, Exercises 10.3.1 - 10.3.3.

Although we've looked at the quantum treatment of identical particles as done by Griffiths, it's worth summarizing Shankar's treatment of the topic as it provides a few more insights.

In classical physics, suppose we have two identical particles, where 'identical' here means that all their physical properties such as mass, size, shape, charge and so on are the same. Suppose we do an experiment in which these two particles collide and rebound in some way. Can we tell which particle ends up in which location? We're not allowed to label the particles by writing on them, for example, since then they would no longer be identical. In classical physics, we can determine which particle is which by tracing their histories. For example, if we start with particle 1 at position $\mathbf{r}_1$ and particle 2 at position $\mathbf{r}_2$, then let them collide, and finally measure their locations at some time after the collision, we might find that one particle ends up at position $\mathbf{r}_3$ and the other at position $\mathbf{r}_4$. If we videoed the collision event, we would see the two particles follow well-defined paths before and after the collision, so by observing which particle followed the path that leads from $\mathbf{r}_1$ to the collision and then out again, we can tell whether it ends up at $\mathbf{r}_3$ or $\mathbf{r}_4$. That is, the identification of a particle depends on our ability to watch it as it travels through space.

In quantum mechanics, because of the uncertainty principle, a particle does not have a well-defined trajectory, since in order to define such a trajectory, we would need to specify its position and momentum precisely at each instant of time as it travels. In terms of our collision experiment, if we measured one particle to be at starting position $\mathbf{r}_1$ at time $t=0$ then we know nothing about its momentum, because we specified the position exactly. Thus we can't tell what trajectory this particle will follow. If we measure the two particles at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ at $t=0$, and then at $\mathbf{r}_3$ and $\mathbf{r}_4$ at some later time, we have no way of knowing which particle ends up at $\mathbf{r}_3$ and which at $\mathbf{r}_4$. In terms of the state vector, this means that the physics in the state vector must be the same if we exchange the two particles within the wave function. Since multiplying a state vector ψ by some complex constant α leaves the physics unchanged, this means that we require

$$\psi(a,b) = \alpha\psi(b,a) \tag{0.1}$$

where a and b represent the two particles.

For a two-particle system, the vector space is spanned by a direct product of the two one-particle vector spaces. Thus the two basis vectors in this vector space that can describe the two particle a and b are $|ab\rangle$ and $|ba\rangle$. If these two particles are identical, then ψ must be some linear combination of these two vectors that satisfies 0.1. That is

$$\psi(b,a) = \beta|ab\rangle + \gamma|ba\rangle \tag{0.2}$$
$$\psi(a,b) = \alpha\psi(b,a) \tag{0.3}$$
$$= \alpha(\beta|ab\rangle + \gamma|ba\rangle) \tag{0.4}$$

However, $\psi(a,b)$ is also just $\psi(b,a)$ with a swapped with b, that is

$$\psi(a,b) = \beta|ba\rangle + \gamma|ab\rangle \tag{0.5}$$

Since $|ab\rangle$ and $|ba\rangle$ are independent, we can equate their coefficients in the last two equations to get

$$\alpha\beta = \gamma \tag{0.6}$$
$$\alpha\gamma = \beta \tag{0.7}$$

Inserting the second equation into the first, we get

$$\alpha^2\gamma = \gamma \tag{0.8}$$
$$\alpha^2 = 1 \tag{0.9}$$
$$\alpha = \pm 1 \tag{0.10}$$

Thus the two possible state functions 0.1 are combinations of $|ab\rangle$ and $|ba\rangle$ such that

$$\psi(a,b) = \pm\psi(b,a) \tag{0.11}$$

The plus sign gives the symmetric state, which can be written as

$$\psi(ab,S) = \frac{1}{\sqrt{2}}(|ab\rangle + |ba\rangle) \tag{0.12}$$

and the minus sign gives the antisymmetric state

$$\psi(ab,A)=\frac{1}{\sqrt{2}}\left(|ab\rangle-|ba\rangle\right) \tag{0.13}$$

The $\frac{1}{\sqrt{2}}$ factor normalizes the states so that

$$\begin{aligned}\langle\psi(ab,S)\,|\,\psi(ab,S)\rangle &= 1 &(0.14)\\ \langle\psi(ab,A)\,|\,\psi(ab,A)\rangle &= 1 &(0.15)\end{aligned}$$

This follows because the basis vectors $|ab\rangle$ and $|ba\rangle$ are orthonormal vectors.

Particles with symmetric states are called *bosons* and particles with antisymmetric states are called *fermions*. The *Pauli exclusion principle* for fermions follows directly from 0.13, since if we set the state variables of the two particles to be the same, that is, $a=b$, then

$$\psi(aa,A)=\frac{1}{\sqrt{2}}\left(|aa\rangle-|aa\rangle\right)=0 \tag{0.16}$$

The symmetry or antisymmetry rules apply to all the properties of the particle taken as an aggregate. That is, the labels a and b can refer to the particle's location plus its other quantum numbers such as spin, charge, and so on. In order for two fermions to be excluded, the states of the two fermions must be identical in all their quantum numbers, so that two fermions with the same orbital location (as two electrons in the same orbital within an atom, for example) are allowed if their spins are different.

Example 1. Suppose we have 2 identical bosons that are measured to be in states $|\phi\rangle$ and $|\chi\rangle$ where $\langle\phi\,|\,\chi\rangle\neq0$. What is their combined state vector? Since they are bosons, their state vector must be symmetric, so we must have

$$\psi(\phi,\chi) = A\,|\phi\chi\rangle+B\,|\chi\phi\rangle \tag{0.17}$$

Because ψ must be symmetric, we must have $A=B$, so that $\psi(\phi,\chi)=\psi(\chi,\phi)$. The 2-particle states can be written as direct products, so we have

$$\psi(\phi,\chi)=A\left(|\phi\rangle\otimes|\chi\rangle+|\chi\rangle\otimes|\phi\rangle\right) \tag{0.18}$$

To normalize, we have, assuming that $|\phi\rangle$ and $|\chi\rangle$ are normalized:

$$(0.19) \qquad |\psi|^2 = 1$$

$$(0.20) \qquad = |A|^2 \left(\langle\phi|\otimes\langle\chi| + \langle\chi|\otimes\langle\phi|\right)\left(|\phi\rangle\otimes|\chi\rangle + |\chi\rangle\otimes|\phi\rangle\right)$$

$$(0.21) \qquad = |A|^2 \left(1 + 1 + |\langle\phi|\chi\rangle|^2 + |\langle\chi|\phi\rangle|^2\right)$$

$$(0.22) \qquad A = \frac{\pm 1}{\sqrt{2\left(1 + |\langle\phi|\chi\rangle|^2\right)}}$$

Thus the normalized state vector is (choosing the + sign):

$$(0.23) \qquad \psi(\phi,\chi) = \frac{1}{\sqrt{2\left(1 + |\langle\phi|\chi\rangle|^2\right)}}\left(|\phi\chi\rangle + |\chi\phi\rangle\right)$$

Notice that this reduces to 0.12 if $\langle\phi|\chi\rangle = 0$.

For more than 2 particles, we need to form state vectors that are either totally symmetric or totally antisymmetric.

Example 2. Suppose we have 3 identical bosons, and they are measured to be in states 3, 3 and 4. Since two of them are in the same state, there are 3 possible combinations, which we can write as $|334\rangle$, $|343\rangle$ and $|433\rangle$. Assuming these states are orthonormal, the full normalized state vector is

$$(0.24) \qquad \psi(3,3,4) = \frac{1}{\sqrt{3}}\left(|334\rangle + |343\rangle + |433\rangle\right)$$

The $\frac{1}{\sqrt{3}}$ ensures that $|\psi(3,3,4)|^2 = 1$.

Incidentally, for $N \geq 3$ particles, it turns out to be impossible to construct a linear combination of the basis states such that the overall state vector is symmetric with respect to the interchange of some pairs of particles and antisymmetric with respect to the interchange of other pairs. A general proof for all N requires group theory, but for $N = 3$ we can show this by brute force. There are $3! = 6$ basis vectors

$$(0.25) \qquad |123\rangle, |231\rangle, |312\rangle, |132\rangle, |321\rangle, |213\rangle$$

Suppose we require the compound state vector to be symmetric with respect to exchanging 1 and 2. We then must have

$$(0.26) \qquad \psi = A\left(|123\rangle + |213\rangle\right) + B\left(|231\rangle + |132\rangle\right) + C\left(|312\rangle + |321\rangle\right)$$

If we now try to make ψ antisymmetric with respect to exchanging 2 and 3, we must have

$$\psi = D(|123\rangle - |132\rangle) + E(|231\rangle - |321\rangle) + F(|312\rangle - |213\rangle) \tag{0.27}$$

Comparing the two, we see that

$$A = D = -F \tag{0.28}$$
$$B = E = -D \tag{0.29}$$
$$C = F = -E \tag{0.30}$$

Eliminating A, B, and C we have, combining the 3 equations:

$$D = -E = F \tag{0.31}$$

But from the first equation, we have $D = -F$, so $F = -F = 0$. From the other equations, this implies that $D = -F = 0$ and $E = -F = 0$, and thus that $A = B = C = 0$. So there is no non-trivial solution that allows both a symmetric and antisymmetric particle exchange within the same state vector.

Example 3. Suppose we have 3 particles and only 3 distinct states that each particle can have. If the particles are distinguishable (not identical) the total number of states is found by considering the possibilities. If all 3 particles are in different states, then there are $3! = 6$ possible overall states. If two particles are in one state and one particle in another, there are $\binom{3}{2} = 3$ ways of choosing the two states, for each of which there are 2 ways of partitioning these two states (that is, which state has 2 particles and which has the other one), and for each of those there are 3 possible configurations, so there are $3 \times 2 \times 3 = 18$ possible configurations. Finally, if all 3 particles are in the same state, there are 3 possibilities. Thus the total for distinguishable particles is $6 + 18 + 3 = 27$.

If the particles are bosons, then if all 3 are in different states, there is only 1 symmetric combination of the 6 basis states. If two particles are in one state and one particle in another, there are $3 \times 2 = 6$ ways of partitioning the states, each of which contributes only one symmetric overall state. Finally, if all 3 particles are in the same state, there are 3 possibilities. Thus the total for bosons is $1 + 6 + 3 = 10$.

For fermions, all three particles must be in different states, so there is only 1 possibility.

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