

## INVARIANCE OF SYMMETRIC AND ANTISYMMETRIC STATES; EXCHANGE OPERATORS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 10, Exercise 10.3.5.

In a system with two particles, the state in the  $X$  basis is given by  $|x_1, x_2\rangle$  where  $x_i$  is the position of particle  $i$ . We can define the exchange operator  $P_{12}$  as an operator that swaps the two particles, so that

$$(0.1) \quad P_{12} |x_1, x_2\rangle = |x_2, x_1\rangle$$

To find the eigenvalues and eigenvectors of  $P_{12}$  we have

$$(0.2) \quad P_{12} |\psi(x_1, x_2)\rangle = \alpha |\psi(x_1, x_2)\rangle = \psi(x_2, x_1)$$

where  $\alpha$  is the eigenvalue and  $|\psi(x_1, x_2)\rangle$  is the eigenvector. Using the same argument as before, we can write

$$(0.3) \quad |\psi(x_1, x_2)\rangle = \beta |x_1, x_2\rangle + \gamma |x_2, x_1\rangle$$

$$(0.4) \quad |\psi(x_2, x_1)\rangle = \beta |x_2, x_1\rangle + \gamma |x_1, x_2\rangle$$

$$(0.5) \quad = \alpha [\beta |x_1, x_2\rangle + \gamma |x_2, x_1\rangle]$$

Equating coefficients in the first and third lines, we arrive at

$$(0.6) \quad \alpha = \pm 1$$

which gives the same symmetric and antisymmetric eigenfunctions that we had before:

$$(0.7) \quad \psi_S(x_1, x_2) = \frac{1}{\sqrt{2}} (|x_1, x_2\rangle + |x_2, x_1\rangle)$$

$$(0.8) \quad \psi_A(x_1, x_2) = \frac{1}{\sqrt{2}} (|x_1, x_2\rangle - |x_2, x_1\rangle)$$

We can derive a couple of other properties of the exchange operator by noting that if it is applied twice in succession, we get the original state back, so that

$$(0.9) \quad P_{12}^2 = I$$

$$(0.10) \quad P_{12} = P_{12}^{-1}$$

Thus the operator is its own inverse.

Consider also the two states  $|x'_1, x'_2\rangle$  and  $|x_1, x_2\rangle$ . Then

$$(0.11) \quad \langle x'_1, x'_2 | P_{12}^\dagger P_{12} | x_1, x_2 \rangle = \langle P_{12} x'_1, x'_2 | P_{12} x_1, x_2 \rangle$$

$$(0.12) \quad = \langle x'_2, x'_1 | x_2, x_1 \rangle$$

$$(0.13) \quad = (\langle x'_2 | \otimes \langle x'_1 |) (|x_2\rangle \otimes |x_1\rangle)$$

$$(0.14) \quad = \delta(x'_2 - x_2) \delta(x'_1 - x_1)$$

However, the last line is just equal to the inner product of the original states, that is

$$(0.15) \quad \langle x'_1, x'_2 | x_1, x_2 \rangle = \delta(x_2 - x'_2) \delta(x_1 - x'_1) = \delta(x'_2 - x_2) \delta(x'_1 - x_1)$$

This means that

$$(0.16) \quad P_{12}^\dagger P_{12} = I$$

$$(0.17) \quad P_{12}^\dagger = P_{12}^{-1} = P_{12}$$

Thus  $P_{12}$  is both Hermitian and unitary.

Shankar asks us to show that, for a general basis vector  $|\omega_1, \omega_2\rangle$ ,  $P_{12} |\omega_1, \omega_2\rangle = |\omega_2, \omega_1\rangle$ . One argument could be that, since the  $X$  basis spans the space, we can express any other vector such as  $|\omega_1, \omega_2\rangle$  as a linear combination of the  $|x_1, x_2\rangle$  vectors, so that applying  $P_{12}$  to  $|\omega_1, \omega_2\rangle$  means applying it to a sum of  $|x_1, x_2\rangle$  vectors, which swaps the two particles in every term. I'm not sure if this is a rigorous result. In any case, if we accept this result it shows that if we start in a state that is totally symmetric (that is, a boson state), this state is an eigenvector of  $P_{12}$  with eigenvalue  $+1$ . Similarly, if we start in an antisymmetric (fermion) state, this state is an eigenvector of  $P_{12}$  with eigenvalue  $-1$ .

Now we can look at some other properties of  $P_{12}$ . Consider

$$(0.18) \quad P_{12} X_1 P_{12} |x_1, x_2\rangle = P_{12} X_1 |x_2, x_1\rangle$$

$$(0.19) \quad = x_2 P_{12} |x_2, x_1\rangle$$

$$(0.20) \quad = x_2 |x_1, x_2\rangle$$

$$(0.21) \quad = X_2 |x_1, x_2\rangle$$

This follows because the operator  $X_1$  operates on the first particle in the state  $|x_2, x_1\rangle$  which on the RHS of the first line is at position  $x_2$ . Thus  $X_1 |x_2, x_1\rangle = x_2 |x_2, x_1\rangle$ , that is,  $X_1$  returns the numerical value of the position of the first particle, which is  $x_2$ . This means that in terms of the operators alone

$$(0.22) \quad P_{12}X_1P_{12} = X_2$$

$$(0.23) \quad P_{12}X_2P_{12} = X_1$$

$$(0.24) \quad P_{12}P_1P_{12} = P_2$$

$$(0.25) \quad P_{12}P_2P_{12} = P_1$$

In the last two lines, the operator  $P_i$  is the momentum of particle  $i$ , and the result follows by applying the operators to the momentum basis state  $|p_1, p_2\rangle$ .

For some general operator which can be expanded in a power series of terms containing powers of  $X_i$  and/or  $P_i$ , we can use 0.10 to insert  $P_{12}P_{12}$  between every factor of  $X_i$  or  $P_i$ . For example

$$(0.26) \quad P_{12}P_1X_2^2X_1P_{12} = P_{12}P_1P_{12}P_{12}X_2P_{12}P_{12}X_2P_{12}P_{12}X_1P_{12}$$

$$(0.27) \quad = P_2X_1^2X_2$$

That is, for any operator  $\Omega(X_1, P_1; X_2, P_2)$  we have

$$(0.28) \quad P_{12}\Omega(X_1, P_1; X_2, P_2)P_{12} = \Omega(X_2, P_2; X_1, P_1)$$

The Hamiltonian for a system of two *identical* particles must be symmetric under exchange of the particles, since it represents an observable (the energy), and this observable must remain unchanged if we swap the particles. (In the case of two fermions, the wave function is antisymmetric, but the wave function itself is not an observable. The wave function gets multiplied by  $-1$  if we swap the particles, but the square modulus of the wave function, which contains the physics, remains the same.) Thus we have

$$(0.29) \quad P_{12}H(X_1, P_1; X_2, P_2)P_{12} = H(X_2, P_2; X_1, P_1) = H(X_1, P_1; X_2, P_2)$$

[Note that this condition doesn't necessarily follow if the two particles are not identical, since exchanging them in this case leads to an observably different system. For example, exchanging the proton and electron in a hydrogen atom leads to a different system.]

The propagator is defined as

$$(0.30) \quad U(t) = e^{-iHt/\hbar}$$

and the propagator dictates how a state evolves according to

$$(0.31) \quad |\psi(t)\rangle = U(t)|\psi(0)\rangle$$

Since the only operator on which  $U$  depends is  $H$ , then  $U$  is also invariant, so that

$$(0.32) \quad P_{12}U(X_1, P_1; X_2, P_2)P_{12} = U(X_2, P_2; X_1, P_1) = U(X_1, P_1; X_2, P_2)$$

Multiplying from the left by  $P_{12}$  and subtracting, we get the commutator

$$(0.33) \quad [U, P_{12}] = 0$$

For a symmetric state  $|\psi_S\rangle$  or antisymmetric state  $|\psi_A\rangle$ , we have

$$(0.34) \quad UP_{12}|\psi_S(0)\rangle = U|\psi_S(0)\rangle = |\psi_S(t)\rangle = P_{12}U|\psi_S(0)\rangle$$

$$(0.35) \quad UP_{12}|\psi_A(0)\rangle = -U|\psi_A(0)\rangle = -|\psi_A(t)\rangle = P_{12}U|\psi_A(0)\rangle$$

This means that states that begin as symmetric or antisymmetric remain symmetric or antisymmetric for all time. In other words, a system that starts in an eigenstate of  $P_{12}$  remains in the same eigenstate as time passes.