TRANSLATIONAL INVARIANCE AND CONSERVATION OF MOMENTUM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 11.

One consequence of the invariance of the Hamiltonian under translation is that the momentum and Hamiltonian commute:

$$[P,H]=0 \tag{1}$$

In quantum mechanics, commuting quantities are simultaneously observable, and we can find a basis for the Hilbert space consisting of eigenstates of both P and H. We've seen that Ehrenfest's theorem allows us to conclude that for such a system, the average momentum is conserved so that $\left\langle \dot{P}\right\rangle =0$. We can go a step further and state that if a system starts out in an eigenstate of P, then it remains in that eigenstate for all time.

First, we need to make a rather subtle observation, which is that

$$[P,H]=0\rightarrow[P,U(t)]=0 \tag{2}$$

That is, if P and H commute, then P also commutes with the propagator $U(t)$. For a time-independent Hamiltonian, the propagator is

$$U(t)=e^{-iHt/\hbar} \tag{3}$$

Since this can be expanded in a power series in the Hamiltonian, condition 2 follows easily enough. What if the Hamiltonian is time-dependent? In this case, the propagator comes out to a time-ordered integral

$$U(t)=T\left\{ \exp\left[-\frac{i}{\hbar}\int_{0}^{t}H\left(t^{\prime}\right)dt^{\prime}\right]\right\} \equiv\lim_{N\rightarrow\infty}\prod_{n=0}^{N-1}e^{-i\Delta H(n\Delta)/\hbar} \tag{4}$$

Here the time interval $[0,t]$ is divided into N time slices, each of length $\Delta=t/N$. As explained in the earlier post, the reason we can't just integrate the RHS directly by summing the exponents is that such a procedure works only if the operators in the exponents all commute with each other. If H is time-dependent, its forms at different times may not commute, so we can't get a simple closed form for $U(t)$.

However, if $[P,H(t)]=0$ for all times, then P commutes with all the exponents on the RHS of 4, so we still get $[P,U(t)]=0$. Another way of looking at this is by imposing the condition $[P,H(t)]=0$ we're saying that if $H(t)$ can be expanded in a power series in X and P, it depends only on P, and not on X. This follows from the fact that

$$[X^n,P]=i\hbar nX^{n-1} \tag{5}$$

so that P does not commute with any power of X.

Given that 2 is valid for all Hamiltonians, then if we start in a eigenstate $|p\rangle$ of P, then

$$\begin{aligned}
P|p\rangle &= p|p\rangle && (6)\\
PU(t)|p\rangle &= U(t)P|p\rangle && (7)\\
&= U(t)p|p\rangle && (8)\\
&= pU(t)|p\rangle && (9)
\end{aligned}$$

Thus $U(t)|p\rangle$ remains an eigenstate of P with the same eigenvalue p for all time. For a single particle moving in one dimension, the state $|p\rangle$ describes a free particle with momentum p (and thus a completely undetermined position).

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