TRANSLATIONAL INVARIANCE IN QUANTUM MECHANICS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 11, Exercises 11.2.1 - 11.2.2.

In classical mechanics, we've seen that if a dynamical variable g is used to generate a transformation of the variables q_i and p_i (the coordinates and canonical momenta), then if the Hamiltonian is invariant under this transformation, the quantity g is conserved, meaning that it remains constant over time. We'd like to extend these results to quantum mechanics, but in doing so, there is one large obstacle. In classical mechanics, we can specify the exact position (given by q_i) and the exact momentum (p_i) at every instant in time for every particle. In other words, every particle has a precisely defined trajectory through phase space. Due to the uncertainty principle, we cannot do this in quantum mechanics, since we cannot specify the position and momentum of any particle with arbitrary precision, so we can't define a precise trajectory for any particle.

The way in which this problem is usually handled is to examine the effects of changes in the expectation values of dynamical variables, rather than with their precise values at any given time. In the case of a single particle moving in one dimension, we can apply this idea to investigate how we might invoke translational invariance. Classically, where x is the position variable and p is the momentum, an infinitesimal translation by a distance ε is given by

$$x \rightarrow x+\varepsilon \tag{0.1}$$

$$p \rightarrow p \tag{0.2}$$

In quantum mechanics, the equivalent translation is reflected in the expectation values:

$$\langle X\rangle \rightarrow \langle X\rangle+\varepsilon \tag{0.3}$$

$$\langle P\rangle \rightarrow \langle P\rangle \tag{0.4}$$

In order to find the expectation values $\langle X\rangle$ and $\langle P\rangle$ we need to use the state vector $|\psi\rangle$. There are two ways of interpreting the transformation. The first, known as the *active transformation picture*, is to say that translating the position generates a new state vector $|\psi_\varepsilon\rangle$ with the properties

$$\langle \psi_\varepsilon | X | \psi_\varepsilon \rangle = \langle \psi | X | \psi \rangle + \varepsilon \quad (0.5)$$

$$\langle \psi_\varepsilon | P | \psi_\varepsilon \rangle = \langle \psi | P | \psi \rangle \quad (0.6)$$

Since $|\psi_\varepsilon\rangle$ is another state vector in the same vector space as $|\psi\rangle$, there must be an operator $T(\varepsilon)$ which we call the translation operator, and which maps one vector onto the other:

$$T(\varepsilon)|\psi\rangle = |\psi_\varepsilon\rangle \quad (0.7)$$

In terms of the translation operator, the translation becomes

$$\left\langle \psi \left| T^\dagger(\varepsilon) X T(\varepsilon) \right| \psi \right\rangle = \langle \psi | X | \psi \rangle + \varepsilon \quad (0.8)$$

$$\left\langle \psi \left| T^\dagger(\varepsilon) P T(\varepsilon) \right| \psi \right\rangle = \langle \psi | P | \psi \rangle \quad (0.9)$$

These relations allow us to define the second interpretation, called the *passive transformation picture*, in which the state vectors do not change, but rather the position and momentum operators change. That is, we can transform the operators according to

$$X \rightarrow T^\dagger(\varepsilon) X T(\varepsilon) = X + \varepsilon I \quad (0.10)$$

$$P \rightarrow T^\dagger(\varepsilon) P T(\varepsilon) = P \quad (0.11)$$

We need to find the explicit form for T. To begin, we consider its effect on a position eigenket $|x\rangle$. One possibility is

$$T(\varepsilon)|x\rangle = |x+\varepsilon\rangle \quad (0.12)$$

However, to be completely general, we should consider the case where T not only shifts x by ε, but also introduces a phase factor. That is, the most general effect of T is

$$T(\varepsilon)|x\rangle = e^{i\varepsilon g(x)/\hbar}|x+\varepsilon\rangle \quad (0.13)$$

where $g(x)$ is some arbitrary real function of x. Using this form, we have, for some arbitrary state vector $|\psi\rangle$:

$$|\psi_\varepsilon\rangle = T(\varepsilon)|\psi\rangle \tag{0.14}$$

$$= T(\varepsilon)\int_{-\infty}^{\infty}|x\rangle\langle x|\psi\rangle\,dx \tag{0.15}$$

$$= \int_{-\infty}^{\infty} e^{i\varepsilon g(x)/\hbar}|x+\varepsilon\rangle\langle x|\psi\rangle\,dx \tag{0.16}$$

$$= \int_{-\infty}^{\infty} e^{i\varepsilon g(x'-\varepsilon)/\hbar}\left|x'\right\rangle\left\langle x'-\varepsilon|\psi\right\rangle dx' \tag{0.17}$$

To get the last line, we changed the integration variable to $x' = x+\varepsilon$. Multiplying by the bra $\langle x|$ gives, using $\langle x|x'\rangle = \delta(x-x')$:

$$\langle x|T(\varepsilon)|\psi\rangle = \langle x|\psi_\varepsilon\rangle = e^{i\varepsilon g(x-\varepsilon)/\hbar}\langle x-\varepsilon|\psi\rangle \tag{0.18}$$

$$= e^{i\varepsilon g(x-\varepsilon)/\hbar}\psi(x-\varepsilon) \tag{0.19}$$

That is, the action of $T(\varepsilon)$ is to move the coordinate axis a distance ε to the right, which means that the new state vector $|\psi_\varepsilon\rangle$ becomes the old state vector at position $x-\varepsilon$. Alternatively, we can leave the coordinate axis alone and shift the wave function a distance ε to the right, so that the new vector at position x is the old vector at position $x-\varepsilon$ (multiplied by a phase factor).

We can now use this result to calculate 0.8 and 0.9:

$$\langle\psi_\varepsilon|X|\psi_\varepsilon\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\psi_\varepsilon|x\rangle\left\langle x|X|x'\right\rangle\left\langle x'|\psi_\varepsilon\right\rangle dx\,dx' \tag{0.20}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\psi_\varepsilon|x\rangle x'\delta\left(x-x'\right)\left\langle x'|\psi_\varepsilon\right\rangle dx\,dx' \tag{0.21}$$

$$= \int_{-\infty}^{\infty}\langle\psi_\varepsilon|x\rangle x\langle x|\psi_\varepsilon\rangle\,dx \tag{0.22}$$

$$= \int_{-\infty}^{\infty} e^{-i\varepsilon g(x-\varepsilon)/\hbar}\psi^*(x-\varepsilon)\,x e^{i\varepsilon g(x-\varepsilon)/\hbar}\psi(x-\varepsilon)\,dx \tag{0.23}$$

$$= \int_{-\infty}^{\infty}\psi^*(x-\varepsilon)\,x\psi(x-\varepsilon)\,dx \tag{0.24}$$

$$= \int_{-\infty}^{\infty}\psi^*\left(x'\right)\left(x'+\varepsilon\right)\psi\left(x'\right)dx' \tag{0.25}$$

$$= \langle\psi|X|\psi\rangle + \varepsilon \tag{0.26}$$

In the second line, we used the matrix element of X

$$\left\langle x\left|X\right|x'\right\rangle = x'\delta\left(x-x'\right) \tag{0.27}$$

and in the penultimate line, we again used the change of integration variable to $x'=x-\varepsilon$. Thus we regain 0.8.

The momentum transforms as follows.

$$\left\langle \psi_{\varepsilon}\left|P\right|\psi_{\varepsilon}\right\rangle = \int_{-\infty}^{\infty}\psi^{*}\left(x-\varepsilon\right)e^{-i\varepsilon g(x-\varepsilon)/\hbar}\left(-i\hbar\frac{d}{dx}\right)\left(e^{i\varepsilon g(x-\varepsilon)/\hbar}\psi\left(x-\varepsilon\right)\right)dx \tag{0.28}$$

$$= \int_{-\infty}^{\infty}\psi^{*}\left(x-\varepsilon\right)\left(\varepsilon\frac{d}{dx}\left(g\left(x-\varepsilon\right)\right)\psi\left(x-\varepsilon\right)-i\hbar\frac{d}{dx}\left(\psi\left(x-\varepsilon\right)\right)\right)dx \tag{0.29}$$

$$= \int_{-\infty}^{\infty}\psi^{*}\left(x'\right)\left(\varepsilon\psi\left(x'\right)\frac{d}{dx'}g\left(x'\right)-i\hbar\frac{d}{dx'}\psi\left(x'\right)\right)dx' \tag{0.30}$$

$$= \varepsilon\left\langle\frac{d}{dx}g\left(x\right)\right\rangle+\left\langle P\right\rangle \tag{0.31}$$

In the third line, we again transformed the integration variable to $x'=x-\varepsilon$, and used the fact that $dx=dx'$, so a derivative with respect to x is the same as a derivative with respect to x'. [This derivation is condensed a bit compared to the derivation of $\left\langle \psi_{\varepsilon}\left|X\right|\psi_{\varepsilon}\right\rangle$, but you can insert a couple of sets of complete states and do the extra integrals if you like.]

If we now impose the condition 0.9 so that the momentum is unchanged by the translation, this is equivalent to choosing the phase function $g\left(x\right)=0$, and this is what is done in most applications.

Having explored the properties of the translation operator, we can now define what we mean by *translational invariance* in quantum mechanics. This is the requirement that the expectation value of the Hamiltonian is unchanged under the transformation. That is

$$\left\langle \psi\left|H\right|\psi\right\rangle = \left\langle \psi_{\varepsilon}\left|H\right|\psi_{\varepsilon}\right\rangle \tag{0.32}$$

For this, we need the explicit form of $T\left(\varepsilon\right)$. Since $\varepsilon=0$ corresponds to no translation, we require $T\left(0\right)=I$. To first order in ε, we can then write

$$T\left(\varepsilon\right)=I-\frac{i\varepsilon}{\hbar}G \tag{0.33}$$

where G is some operator, called the generator of translations, that is to be determined. From 0.13 (with $g=0$ from now on), we have

(0.34) $$\left\langle x'+\varepsilon\left|x+\varepsilon\right.\right\rangle=\left\langle x'\left|T^{\dagger}\left(\varepsilon\right)T\left(\varepsilon\right)\right|x\right\rangle=\delta\left(x'-x\right)=\left\langle x'\left|x\right.\right\rangle$$

so we must have

(0.35) $$T^{\dagger}\left(\varepsilon\right)T\left(\varepsilon\right)=I$$

so that T is unitary. Applying this condition to 0.33 up to order ε, we have

(0.36) $$T^{\dagger}\left(\varepsilon\right)T\left(\varepsilon\right) = \left(I+\frac{i\varepsilon}{\hbar}G^{\dagger}\right)\left(I-\frac{i\varepsilon}{\hbar}G\right)$$

(0.37) $$= I+\frac{i\varepsilon}{\hbar}\left(G^{\dagger}-G\right)+\mathscr{O}\left(\varepsilon^{2}\right)$$

Requiring 0.35 shows that $G=G^{\dagger}$ so G is Hermitian. Now, from 0.19 ($g=0$ again) we have

(0.38) $$\left\langle x\left|T\left(\varepsilon\right)\right|\psi\right\rangle=\psi\left(x-\varepsilon\right)$$

We expand both sides to order ε:

(0.39) $$\left\langle x\left|I\right|\psi\right\rangle-\frac{i\varepsilon}{\hbar}\left\langle x\left|G\right|\psi\right\rangle=\psi\left(x\right)-\varepsilon\frac{d\psi}{dx}$$

Since $\left\langle x\left|I\right|\psi\right\rangle=\left\langle x\left|\psi\right.\right\rangle=\psi\left(x\right)$, we have

(0.40) $$\left\langle x\left|G\right|\psi\right\rangle=-i\hbar\frac{d\psi}{dx}=\left\langle x\left|P\right|\psi\right\rangle$$

so $G=P$ and the momentum operator is the generator of translations, and the translation operator is, to order ε

(0.41) $$T\left(\varepsilon\right)=I-\frac{i\varepsilon}{\hbar}P$$

By plugging this into 0.32 and expanding the RHS, we find that in order for the Hamiltonian to be invariant, the expectation value of the commutator $\left[P,H\right]$ must be zero (the derivation is done in Shankar's eqn 11.2.15). Using Ehrenfest's theorem we then find that the expectation value $\left\langle\dot{P}\right\rangle=\left\langle\left[P,H\right]\right\rangle=0$, so that the expectation value of P is conserved over time.

Note that we *cannot* say that the momentum itself (rather than just its expectation value) is conserved since, due to the uncertainty principle, we never know what the exact momentum is at any given time.

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