TIME TRANSLATION AND CONSERVATION OF ENERGY

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 11, Section 11.3.

We can investigate the effect of a quantum system being invariant under time translation by considering the evolution of a state vector using a propagator. A state at time t is given in terms of the state at time $t=0$ according to

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = e^{-itH/\hbar}|\psi(0)\rangle \tag{1}$$

Strictly speaking, this equation is true only if H is time-independent, since in the time-dependent case, we need to express the propagator as a time-ordered integral. However, if we let the system evolve for only an infinitesimal time ε, we can ignore the complexities of the time-ordered integral and write, to first order in ε

$$U(\varepsilon) = e^{-i\varepsilon H(0)/\hbar} = I - \frac{i\varepsilon H(0)}{\hbar} \tag{2}$$

Note that it doesn't matter if we use the value of H at time $t=0$ or $t=\varepsilon$ or at some time in between, since the differences between these values are of order ε, and thus make no difference to $U(\varepsilon)$ to first order in ε.

Now suppose we prepare the same system (which we'll call $|\psi_0\rangle$) at some time $t=t_1$ and consider how the system evolves over an infinitesimal time ε starting from $t=t_1$. We then have

$$|\psi(t_1+\varepsilon)\rangle = U(t_1+\varepsilon)|\psi_0\rangle \tag{3}$$

$$= \left(I - \frac{i\varepsilon H(t_1)}{\hbar}\right)|\psi_0\rangle \tag{4}$$

The idea behind time translation invariance is that it shouldn't make any difference at what time we prepare a system, provided that the system is prepared identically at whatever time we actually do prepare it. In other words, if we had prepared our system above at $t=t_2$ instead of $t=t_1$ and then let it evolve for an infinitesimal time ε, we should end up with exactly the same state. That is, we require that

$$|\psi(t_2+\varepsilon)\rangle = \left(I - \frac{i\varepsilon H(t_2)}{\hbar}\right)|\psi_0\rangle \tag{5}$$
$$= \left(I - \frac{i\varepsilon H(t_1)}{\hbar}\right)|\psi_0\rangle \tag{6}$$

Rearranging things, we get

$$-\frac{i\varepsilon}{\hbar}\left(H(t_2) - H(t_1)\right)|\psi_0\rangle = 0 \tag{7}$$

The initial state can be anything we like, so in order for this condition to be always true, we must have

$$H(t_2) = H(t_1) \tag{8}$$

Again, the two times t_1 and t_2 at which we prepared the system are arbitrary (and not necessarily separated by an infinitesimal time, so they could be years apart), so this condition implies that H itself must be constant in time. For a time-independent operator A, Ehrenfest's theorem says that

$$\langle \dot{A} \rangle = -\frac{i}{\hbar}\langle [A,H] \rangle \tag{9}$$

If $A = H$, then the commutator is $[H,H] = 0$, so time translation invariance implies that

$$\langle \dot{H} \rangle = 0 \tag{10}$$

That is, time translation invariance implies that the average energy of the system is conserved.

Clearly, energy is conserved if the system is in an energy eigenstate, since then the energy has a single, unchanging value. However, if we prepare the state as a combination of energy eigenstates, then the system has the form

$$\psi(x,t) = \sum_k c_k e^{-iE_k t/\hbar}\psi_k(x) \tag{11}$$

where the c_k are constant coefficients. A measurement of the energy on such a system can yield any of the energies E_k for which $c_k \neq 0$, so it might seem that we're violating the conservation of energy. The point is that, *on average*, the energy is

$$\langle E \rangle = \sum_k |c_k|^2 E_k \tag{12}$$

and it is this average that doesn't change with time. In dealing with averages, we're also retaining consistency with the infamous energy-time uncertainty relation.