TRANSLATION INVARIANCE IN TWO DIMENSIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.1.1.

In preparation for an examination of rotation invariance, we'll have a look at translational invariance in two dimensions. We can apply much of what we did with translation in one dimension, where we showed that the momentum P is the generator of translations. In particular, the translation operator $T(\varepsilon)$ for an infinitesimal translation ε is

$$T(\varepsilon)=I-\frac{i\varepsilon}{\hbar}P \tag{1}$$

In two dimensions, we can write an infinitesimal translation as δa where

$$\delta a=\delta a_x\hat{\mathbf{x}}+\delta a_y\hat{\mathbf{y}} \tag{2}$$

In one dimension, we showed earlier that

$$\langle x|T(\varepsilon)|\psi\rangle=\psi(x-\varepsilon) \tag{3}$$

The analogous relation in two dimensions is

$$\langle x,y|T(\delta a)|\psi\rangle=\psi(x-\delta a_x,y-\delta a_y) \tag{4}$$

We can verify that the correct form for $T(\delta a)$ is

$$\begin{aligned}T(\delta a) &= I-\frac{i}{\hbar}\delta a\cdot\mathbf{P} &(5)\\ &= I-\frac{i}{\hbar}(\delta a_xP_x+\delta a_yP_y) &(6)\end{aligned}$$

Using the representation of momentum in the position basis, which is

$$\begin{aligned}P_x &= -i\hbar\frac{\partial}{\partial x} &(7)\\ P_y &= -i\hbar\frac{\partial}{\partial y} &(8)\end{aligned}$$

the LHS of 4 is, using $\langle x,y|\psi\rangle=\psi(x,y)$:

$$\langle x,y\,|T(\delta a)|\,\psi\rangle = \left\langle x,y\left|I-\frac{i}{\hbar}(\delta a_x P_x+\delta a_y P_y)\right|\psi\right\rangle \tag{9}$$

$$= \psi(x,y)-\delta a_x\frac{\partial\psi}{\partial x}-\delta a_y\frac{\partial\psi}{\partial y} \tag{10}$$

The last line is also what we get if we expand the RHS of 4 to first order in δa, which verifies that 5 is correct, so that the two-dimensional momentum **P** is the generator of two-dimensional translations.

We can apply the exponentiation technique we used in the one-dimensional case to obtain the translation operator for a finite translation in two dimensions. We need to be careful that we don't run into problems with non-commuting operators, but in view of 7 and 8 and the fact that derivatives with respect to different independent variables commute, we see that

$$[P_x,P_y]=0 \tag{11}$$

We can divide a finite translation **a** into N small steps, each of size $\frac{\mathbf{a}}{N}$, so that the translation is

$$T(\mathbf{a}) = \left(I-\frac{i}{\hbar N}\mathbf{a}\cdot\mathbf{P}\right)^N \tag{12}$$

Because the two components of momentum commute, we can take the limit of this expression to get the exponential form:

$$T(\mathbf{a}) = \lim_{N\to\infty}\left(I-\frac{i}{\hbar N}\mathbf{a}\cdot\mathbf{P}\right)^N = e^{-i\mathbf{a}\cdot\mathbf{P}/\hbar} \tag{13}$$

Again, because the two components of momentum commute, we can combine two translations, by **a** and then by **b**, to get

$$T(\mathbf{b})\,T(\mathbf{a}) = e^{-i\mathbf{b}\cdot\mathbf{P}/\hbar}e^{-i\mathbf{a}\cdot\mathbf{P}/\hbar} = e^{-i(\mathbf{a}+\mathbf{b})\cdot\mathbf{P}/\hbar} = T(\mathbf{b}+\mathbf{a}) \tag{14}$$

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