

## TRANSLATION INVARIANCE IN TWO DIMENSIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.1.1.

In preparation for an examination of rotation invariance, we'll have a look at translational invariance in two dimensions. We can apply much of what we did with translation in one dimension, where we showed that the momentum  $P$  is the generator of translations. In particular, the translation operator  $T(\epsilon)$  for an infinitesimal translation  $\epsilon$  is

$$(0.1) \quad T(\epsilon) = I - \frac{i\epsilon}{\hbar}P$$

In two dimensions, we can write an infinitesimal translation as  $\delta a$  where

$$(0.2) \quad \delta a = \delta a_x \hat{x} + \delta a_y \hat{y}$$

In one dimension, we showed earlier that

$$(0.3) \quad \langle x | T(\epsilon) | \psi \rangle = \psi(x - \epsilon)$$

The analogous relation in two dimensions is

$$(0.4) \quad \langle x, y | T(\delta a) | \psi \rangle = \psi(x - \delta a_x, y - \delta a_y)$$

We can verify that the correct form for  $T(\delta a)$  is

$$(0.5) \quad T(\delta a) = I - \frac{i}{\hbar} \delta a \cdot \mathbf{P}$$

$$(0.6) \quad = I - \frac{i}{\hbar} (\delta a_x P_x + \delta a_y P_y)$$

Using the representation of momentum in the position basis, which is

$$(0.7) \quad P_x = -i\hbar \frac{\partial}{\partial x}$$

$$(0.8) \quad P_y = -i\hbar \frac{\partial}{\partial y}$$

the LHS of 0.4 is, using  $\langle x, y | \psi \rangle = \psi(x, y)$ :

$$(0.9) \quad \langle x, y | T(\delta a) | \psi \rangle = \left\langle x, y \left| I - \frac{i}{\hbar} (\delta a_x P_x + \delta a_y P_y) \right| \psi \right\rangle$$

$$(0.10) \quad = \psi(x, y) - \delta a_x \frac{\partial \psi}{\partial x} - \delta a_y \frac{\partial \psi}{\partial y}$$

The last line is also what we get if we expand the RHS of 0.4 to first order in  $\delta a$ , which verifies that 0.5 is correct, so that the two-dimensional momentum  $\mathbf{P}$  is the generator of two-dimensional translations.

We can apply the exponentiation technique we used in the one-dimensional case to obtain the translation operator for a finite translation in two dimensions. We need to be careful that we don't run into problems with non-commuting operators, but in view of 0.7 and 0.8 and the fact that derivatives with respect to different independent variables commute, we see that

$$(0.11) \quad [P_x, P_y] = 0$$

We can divide a finite translation  $\mathbf{a}$  into  $N$  small steps, each of size  $\frac{\mathbf{a}}{N}$ , so that the translation is

$$(0.12) \quad T(\mathbf{a}) = \left( I - \frac{i}{\hbar N} \mathbf{a} \cdot \mathbf{P} \right)^N$$

Because the two components of momentum commute, we can take the limit of this expression to get the exponential form:

$$(0.13) \quad T(\mathbf{a}) = \lim_{N \rightarrow \infty} \left( I - \frac{i}{\hbar N} \mathbf{a} \cdot \mathbf{P} \right)^N = e^{-i\mathbf{a} \cdot \mathbf{P} / \hbar}$$

Again, because the two components of momentum commute, we can combine two translations, by  $\mathbf{a}$  and then by  $\mathbf{b}$ , to get

$$(0.14) \quad T(\mathbf{b}) T(\mathbf{a}) = e^{-i\mathbf{b} \cdot \mathbf{P} / \hbar} e^{-i\mathbf{a} \cdot \mathbf{P} / \hbar} = e^{-i(\mathbf{a} + \mathbf{b}) \cdot \mathbf{P} / \hbar} = T(\mathbf{b} + \mathbf{a})$$

#### PINGBACKS

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