ROTATIONAL INVARIANCE IN TWO DIMENSIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.2.1.

As a first look at rotational invariance in quantum mechanics, we'll look at two-dimensional rotations about the z axis. Classically, a rotation by an angle ϕ_0 about the z axis is given by the matrix equation for the coordinates

$$\left[\begin{array}{c}\bar{x}\\ \bar{y}\end{array}\right]=\left[\begin{array}{cc}\cos\phi_0 & -\sin\phi_0\\ \sin\phi_0 & \cos\phi_0\end{array}\right]\left[\begin{array}{c}x\\ y\end{array}\right] \tag{1}$$

The momenta transform the same way, since we are merely changing the direction of the x and y axes. Thus we have also

$$\left[\begin{array}{c}\bar{p}_x\\ \bar{p}_y\end{array}\right]=\left[\begin{array}{cc}\cos\phi_0 & -\sin\phi_0\\ \sin\phi_0 & \cos\phi_0\end{array}\right]\left[\begin{array}{c}p_x\\ p_y\end{array}\right] \tag{2}$$

The rotation matrix can be written as an operator, defined as

$$R(\phi_0\hat{\mathbf{z}})=\left[\begin{array}{cc}\cos\phi_0 & -\sin\phi_0\\ \sin\phi_0 & \cos\phi_0\end{array}\right] \tag{3}$$

In quantum mechanics, due to the uncertainty principle we cannot specify position and momentum precisely at the same time, so as with the case of translational invariance, we deal with expectation values. As usual, a rotation is represented by a unitary operator $U[R(\phi_0\hat{\mathbf{z}})]$ so that a quantum state transforms according to

$$|\psi\rangle\rightarrow|\psi_R\rangle=U[R]|\psi\rangle \tag{4}$$

Dealing with expectation values means that the rotation operator must satisfy

$$\langle X\rangle_R=\langle X\rangle\cos\phi_0-\langle Y\rangle\sin\phi_0 \tag{5}$$
$$\langle Y\rangle_R=\langle X\rangle\sin\phi_0+\langle Y\rangle\cos\phi_0 \tag{6}$$
$$\langle P_x\rangle_R=\langle P_x\rangle\cos\phi_0-\langle P_y\rangle\sin\phi_0 \tag{7}$$
$$\langle P_y\rangle_R=\langle P_x\rangle\sin\phi_0+\langle P_y\rangle\cos\phi_0 \tag{8}$$

The expectation values on the LHS of these equations are calculated using the rotated state, so that

$$\langle X\rangle_R = \langle\psi_R|X|\,\psi_R\rangle \tag{9}$$

and so on.

In two dimensions, the position eigenkets depend on the two independent coordinates x and y, and each of these eigenkets transforms under rotation in the same way the position variables above. Operating on such an eigenket with the unitary rotation operator thus must give

$$U[R]\,|x,y\rangle = |x\cos\phi_0 - y\sin\phi_0, x\sin\phi_0 + y\cos\phi_0\rangle \tag{10}$$

As with the translation operator, we try to construct an explicity form for $U[R]$ by considering an infinitesimal rotation $\varepsilon_z\hat{\mathbf{z}}$ about the z axis. We propose that the unitary operator for this rotation is given by

$$U[R(\varepsilon_z\hat{\mathbf{z}})] = I - \frac{i\varepsilon_z L_z}{\hbar} \tag{11}$$

where L_z is, at this stage, an unknown operator called the generator of infinitesimal rotations (although, as the notation suggests, it will turn out to be the z component of angular momentum). Under this rotation, we have, to first order in ε_z:

$$U[R(\varepsilon_z\hat{\mathbf{z}})]\,|x,y\rangle = |x - y\varepsilon_z, x\varepsilon_z + y\rangle \tag{12}$$

Note that we've omitted a possible phase factor in this rotation. That is, we could have written

$$U[R(\varepsilon_z\hat{\mathbf{z}})]\,|x,y\rangle = e^{i\varepsilon_z g(x,y)/\hbar}\,|x - y\varepsilon_z, x\varepsilon_z + y\rangle \tag{13}$$

for some real function $g(x,y)$. Dropping the phase factor has the effect of making the momentum expectation values transform in the same way as the position expectaton values, as shown by Shankar in his equation 12.2.13, so we'll just take the phase factor to be 1 from now on.

We can now find the position space form of a general state vector $|\psi\rangle$ under an infinitesimal rotation by following a similar procedure to that for a translation.

We have

$$\left|\psi_{\varepsilon_z}\right\rangle = U\left[R\left(\varepsilon_z\hat{\mathbf{z}}\right)\right]\left|\psi\right\rangle \tag{14}$$

$$= U\left[R\left(\varepsilon_z\hat{\mathbf{z}}\right)\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|x,y\right\rangle\left\langle x,y\left|\psi\right.\right\rangle dx\,dy \tag{15}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U\left[R\left(\varepsilon_z\hat{\mathbf{z}}\right)\right]\left|x,y\right\rangle\left\langle x,y\left|\psi\right.\right\rangle dx\,dy \tag{16}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|x-y\varepsilon_z, x\varepsilon_z+y\right\rangle\left\langle x,y\left|\psi\right.\right\rangle dx\,dy \tag{17}$$

We can now change integration variables if we define

$$x' \equiv x - y\varepsilon_z \tag{18}$$

$$y' = x\varepsilon_z + y \tag{19}$$

The differentials transform by considering terms only up to first order in infinitesimal quantities, so we have

$$dx' = dx - \varepsilon_z dy = dx \tag{20}$$

$$dy' = \varepsilon_z dx + dy = dy \tag{21}$$

Also, to first order in infinitesimal quantities, we can invert the variables to get

$$x' + \varepsilon_z y' = x - y\varepsilon_z + x\varepsilon_z^2 + y\varepsilon_z = x \tag{22}$$

$$y' - \varepsilon_z x' = x\varepsilon_z + y - x\varepsilon_z + y\varepsilon_z^2 = y \tag{23}$$

The ranges of integration are still $\pm\infty$, so we end up with

$$\left|\psi_{\varepsilon_z}\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|x',y'\right\rangle\left\langle x'+\varepsilon_z y', y'-\varepsilon_z x'\left|\psi\right.\right\rangle dx'\,dy' \tag{24}$$

Multiplying on the left by the bra $\langle x,y|$ we have

(25)

$$\left\langle x,y\left|\psi_{\varepsilon_z}\right.\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle x,y\left|x',y'\right.\right\rangle\left\langle x'+\varepsilon_z y', y'-\varepsilon_z x'\left|\psi\right.\right\rangle dx'\,dy'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta\left(x-x'\right)\delta\left(y-y'\right)\left\langle x'+\varepsilon_z y', y'-\varepsilon_z x'\left|\psi\right.\right\rangle dx'\,dy' \tag{26}$$

$$= \left\langle x+\varepsilon_z y, y-\varepsilon_z x\left|\psi\right.\right\rangle \tag{27}$$

$$= \psi\left(x+\varepsilon_z y, y-\varepsilon_z x\right) \tag{28}$$

This can now be expanded in a 2-variable Taylor series to give, to first order in ε_z:

$$\psi(x+\varepsilon_z y, y-\varepsilon_z x)=\psi(x,y)+y\varepsilon_z\frac{\partial\psi}{\partial x}-x\varepsilon_z\frac{\partial\psi}{\partial y} \tag{29}$$

We can compare this with 11 inserted into 14:

$$\left\langle x,y\middle|\psi_{\varepsilon_z}\right\rangle = \langle x,y|U[R(\varepsilon_z\hat{\mathbf{z}})]|\psi\rangle \tag{30}$$

$$= \left\langle x,y\middle|I-\frac{i\varepsilon_z L_z}{\hbar}\middle|\psi\right\rangle \tag{31}$$

$$= \psi(x,y)-\frac{i\varepsilon_z}{\hbar}\langle x,y|L_z|\psi\rangle \tag{32}$$

Setting 32 equal to 29 we have

$$-\frac{i\varepsilon_z}{\hbar}\langle x,y|L_z|\psi\rangle = y\varepsilon_z\frac{\partial\psi}{\partial x}-x\varepsilon_z\frac{\partial\psi}{\partial y} \tag{33}$$

$$\langle x,y|L_z|\psi\rangle = x\left(-i\hbar\frac{\partial\psi}{\partial y}\right)-y\left(-i\hbar\frac{\partial\psi}{\partial x}\right) \tag{34}$$

Using the position-space forms of the momenta

$$P_x = -i\hbar\frac{\partial}{\partial x} \tag{35}$$

$$P_y = -i\hbar\frac{\partial}{\partial y} \tag{36}$$

we see that L_z is given by

$$L_z = XP_y - YP_x \tag{37}$$

which is the quantum equivalent of the z component of angular momentum, as promised.

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