ROTATIONAL TRANSFORMATIONS USING PASSIVE TRANSFORMATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.2.2.

We can also derive the generator of rotations L_z by considering passive transformations of the position and momentum operators, in a way similar to that used for deriving the generator of translations. In a passive transformation, the operators are modified while the state vectors remain the same. For an infinitesimal rotation $\varepsilon_z\hat{\mathbf{z}}$ about the z axis in two dimensions, the unitary operator has the form

$$U\left[R\left(\varepsilon_z\hat{\mathbf{z}}\right)\right]=I-\frac{i\varepsilon_z L_z}{\hbar} \tag{1}$$

For a finite rotation by $\phi_0\hat{\mathbf{z}}$ the transformations are given by

$$\left\langle X\right\rangle_R=\left\langle X\right\rangle\cos\phi_0-\left\langle Y\right\rangle\sin\phi_0 \tag{2}$$
$$\left\langle Y\right\rangle_R=\left\langle X\right\rangle\sin\phi_0+\left\langle Y\right\rangle\cos\phi_0 \tag{3}$$
$$\left\langle P_x\right\rangle_R=\left\langle P_x\right\rangle\cos\phi_0-\left\langle P_y\right\rangle\sin\phi_0 \tag{4}$$
$$\left\langle P_y\right\rangle_R=\left\langle P_x\right\rangle\sin\phi_0+\left\langle P_y\right\rangle\cos\phi_0 \tag{5}$$

For the infinitesimal transformation, $\phi_0=\varepsilon_z$ and these equations reduce to

$$\left\langle X\right\rangle_R = \left\langle X\right\rangle-\left\langle Y\right\rangle\varepsilon_z \tag{6}$$
$$\left\langle Y\right\rangle_R = \left\langle X\right\rangle\varepsilon_z+\left\langle Y\right\rangle \tag{7}$$
$$\left\langle P_x\right\rangle_R = \left\langle P_x\right\rangle-\left\langle P_y\right\rangle\varepsilon_z \tag{8}$$
$$\left\langle P_y\right\rangle_R = \left\langle P_x\right\rangle\varepsilon_z+\left\langle P_y\right\rangle \tag{9}$$

In the passive transformation scheme, we move the transformation to the operators to get

$$\begin{aligned}
U^{\dagger}[R]XU[R] &= X-Y\varepsilon_z && (10)\\
U^{\dagger}[R]YU[R] &= X\varepsilon_z+Y && (11)\\
U^{\dagger}[R]P_xU[R] &= P_x-P_y\varepsilon_z && (12)\\
U^{\dagger}[R]P_yU[R] &= P_x\varepsilon_z+P_y && (13)
\end{aligned}$$

Substituting 1 into these equations gives us the commutation relations satisfied by L_z. For example, in the first equation we have

$$\begin{aligned}
U^{\dagger}[R]XU[R] &= \left(I+\frac{i\varepsilon_z L_z}{\hbar}\right)X\left(I-\frac{i\varepsilon_z L_z}{\hbar}\right) && (14)\\
&= X+\frac{i\varepsilon_z}{\hbar}(L_zX-XL_z) && (15)\\
&= X-Y\varepsilon_z && (16)
\end{aligned}$$

Equating the last two lines, we get

$$[X,L_z]=-i\hbar Y \tag{17}$$

Similarly, for the other three equations we get

$$\begin{aligned}
[Y,L_z] &= i\hbar X && (18)\\
[P_x,L_z] &= -i\hbar P_y && (19)\\
[P_y,L_z] &= i\hbar P_x && (20)
\end{aligned}$$

We can use these commutation relations to derive the form of L_z by using the commutation relations for coordinates and momenta:

$$[X,P_x]=[Y,P_y]=i\hbar \tag{21}$$

with all other commutators involving X,Y,P_x and P_y being zero. Starting with 17, we see that

$$[X,L_z]=-[X,P_x]Y \tag{22}$$

We can therefore deduce that

$$L_z=-P_xY+f(X,Y,P_y) \tag{23}$$

where f is some unknown function. We must include f since the commutators of X with X,Y and P_y are all zero, so adding on f still satisfies 17. (You can think of it as similar to adding on the constant in an indefinite integral.)

Now from 18, we have

$$[Y,L_z]=[Y,P_y]X \tag{24}$$

so combining this with 23 we have

$$L_z=-P_xY+P_yX+g(X,Y) \tag{25}$$

The undetermined function is now a function only of X and Y, since the dependence of L_z on P_x and P_y has been determined uniquely by the commutators 17 and 18.

From 19 we have

$$[P_x,L_z]=[P_x,X]P_y \tag{26}$$

We can see that this is satisfied already by 25, except that we now know that the function g cannot depend on X, since then $[P_x,g]\neq 0$. Thus we have narrowed down L_z to

$$L_z=-P_xY+P_yX+h(Y) \tag{27}$$

Finally, from 20 we have

$$[P_y,L_z]=-[P_y,Y]P_x \tag{28}$$

This is satisfied by 27 if we take $h=0$ (well, technically, we could take h to be some constant, but we might as well take the constant to be zero), giving us the final form for L_z:

$$L_z=-P_xY+P_yX \tag{29}$$

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