# ROTATIONAL TRANSFORMATIONS USING PASSIVE TRANSFORMATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.2.2.

We can also derive the generator of rotations $L_z$ by considering passive transformations of the position and momentum operators, in a way similar to that used for deriving the generator of translations. In a passive transformation, the operators are modified while the state vectors remain the same. For an infinitesimal rotation $\varepsilon_z\hat{\mathbf{z}}$ about the $z$ axis in two dimensions, the unitary operator has the form

$$U\left[R\left(\varepsilon_z\hat{\mathbf{z}}\right)\right]=I-\frac{i\varepsilon_z L_z}{\hbar} \tag{1}$$

For a finite rotation by $\phi_0\hat{\mathbf{z}}$ the transformations are given by

$$\langle X\rangle_R = \langle X\rangle\cos\phi_0-\langle Y\rangle\sin\phi_0 \tag{2}$$
$$\langle Y\rangle_R = \langle X\rangle\sin\phi_0+\langle Y\rangle\cos\phi_0 \tag{3}$$
$$\langle P_x\rangle_R = \langle P_x\rangle\cos\phi_0-\langle P_y\rangle\sin\phi_0 \tag{4}$$
$$\langle P_y\rangle_R = \langle P_x\rangle\sin\phi_0+\langle P_y\rangle\cos\phi_0 \tag{5}$$

For the infinitesimal transformation, $\phi_0=\varepsilon_z$ and these equations reduce to

$$\langle X\rangle_R = \langle X\rangle-\langle Y\rangle\varepsilon_z \tag{6}$$
$$\langle Y\rangle_R = \langle X\rangle\varepsilon_z+\langle Y\rangle \tag{7}$$
$$\langle P_x\rangle_R = \langle P_x\rangle-\langle P_y\rangle\varepsilon_z \tag{8}$$
$$\langle P_y\rangle_R = \langle P_x\rangle\varepsilon_z+\langle P_y\rangle \tag{9}$$

In the passive transformation scheme, we move the transformation to the operators to get

$$\begin{aligned} U^{\dagger}[R]XU[R] &= X - Y\varepsilon_z & (10)\\ U^{\dagger}[R]YU[R] &= X\varepsilon_z + Y & (11)\\ U^{\dagger}[R]P_xU[R] &= P_x - P_y\varepsilon_z & (12)\\ U^{\dagger}[R]P_yU[R] &= P_x\varepsilon_z + P_y & (13) \end{aligned}$$

Substituting 1 into these equations gives us the commutation relations satisfied by $L_z$. For example, in the first equation we have

$$U^{\dagger}[R]XU[R] = \left(I + \frac{i\varepsilon_z L_z}{\hbar}\right) X \left(I - \frac{i\varepsilon_z L_z}{\hbar}\right) \tag{14}$$

$$= X + \frac{i\varepsilon_z}{\hbar}\left(L_z X - X L_z\right) \tag{15}$$

$$= X - Y\varepsilon_z \tag{16}$$

Equating the last two lines, we get

$$[X, L_z] = -i\hbar Y \tag{17}$$

Similarly, for the other three equations we get

$$\begin{aligned} [Y, L_z] &= i\hbar X & (18)\\ [P_x, L_z] &= -i\hbar P_y & (19)\\ [P_y, L_z] &= i\hbar P_x & (20) \end{aligned}$$

We can use these commutation relations to derive the form of $L_z$ by using the commutation relations for coordinates and momenta:

$$[X, P_x] = [Y, P_y] = i\hbar \tag{21}$$

with all other commutators involving $X, Y, P_x$ and $P_y$ being zero. Starting with 17, we see that

$$[X, L_z] = -[X, P_x]Y \tag{22}$$

We can therefore deduce that

$$L_z = -P_x Y + f(X, Y, P_y) \tag{23}$$

where $f$ is some unknown function. We must include $f$ since the commutators of $X$ with $X, Y$ and $P_y$ are all zero, so adding on $f$ still satisfies 17. (You can think of it as similar to adding on the constant in an indefinite integral.)

Now from 18, we have

$$[Y,L_z] = [Y,P_y]\,X \tag{24}$$

so combining this with 23 we have

$$L_z = -P_xY + P_yX + g(X,Y) \tag{25}$$

The undetermined function is now a function only of $X$ and $Y$, since the dependence of $L_z$ on $P_x$ and $P_y$ has been determined uniquely by the commutators 17 and 18.

From 19 we have

$$[P_x,L_z] = [P_x,X]\,P_y \tag{26}$$

We can see that this is satisfied already by 25, except that we now know that the function $g$ cannot depend on $X$, since then $[P_x,g] \neq 0$. Thus we have narrowed down $L_z$ to

$$L_z = -P_xY + P_yX + h(Y) \tag{27}$$

Finally, from 20 we have

$$[P_y,L_z] = -[P_y,Y]\,P_x \tag{28}$$

This is satisfied by 27 if we take $h = 0$ (well, technically, we could take $h$ to be some constant, but we might as well take the constant to be zero), giving us the final form for $L_z$:

$$L_z = -P_xY + P_yX \tag{29}$$

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