

ROTATIONAL TRANSFORMATIONS USING PASSIVE TRANSFORMATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.2.2.

We can also derive the generator of rotations L_z by considering passive transformations of the position and momentum operators, in a way similar to that used for deriving the generator of translations. In a passive transformation, the operators are modified while the state vectors remain the same. For an infinitesimal rotation $\epsilon_z \hat{\mathbf{z}}$ about the z axis in two dimensions, the unitary operator has the form

$$(0.1) \quad U [R (\epsilon_z \hat{\mathbf{z}})] = I - \frac{i\epsilon_z L_z}{\hbar}$$

For a finite rotation by $\phi_0 \hat{\mathbf{z}}$ the transformations are given by

$$(0.2) \quad \langle X \rangle_R = \langle X \rangle \cos \phi_0 - \langle Y \rangle \sin \phi_0$$

$$(0.3) \quad \langle Y \rangle_R = \langle X \rangle \sin \phi_0 + \langle Y \rangle \cos \phi_0$$

$$(0.4) \quad \langle P_x \rangle_R = \langle P_x \rangle \cos \phi_0 - \langle P_y \rangle \sin \phi_0$$

$$(0.5) \quad \langle P_y \rangle_R = \langle P_x \rangle \sin \phi_0 + \langle P_y \rangle \cos \phi_0$$

For the infinitesimal transformation, $\phi_0 = \epsilon_z$ and these equations reduce to

$$(0.6) \quad \langle X \rangle_R = \langle X \rangle - \langle Y \rangle \epsilon_z$$

$$(0.7) \quad \langle Y \rangle_R = \langle X \rangle \epsilon_z + \langle Y \rangle$$

$$(0.8) \quad \langle P_x \rangle_R = \langle P_x \rangle - \langle P_y \rangle \epsilon_z$$

$$(0.9) \quad \langle P_y \rangle_R = \langle P_x \rangle \epsilon_z + \langle P_y \rangle$$

In the passive transformation scheme, we move the transformation to the operators to get

$$(0.10) \quad U^\dagger [R] X U [R] = X - Y \epsilon_z$$

$$(0.11) \quad U^\dagger [R] Y U [R] = X \epsilon_z + Y$$

$$(0.12) \quad U^\dagger [R] P_x U [R] = P_x - P_y \epsilon_z$$

$$(0.13) \quad U^\dagger [R] P_y U [R] = P_x \epsilon_z + P_y$$

Substituting 0.1 into these equations gives us the commutation relations satisfied by L_z . For example, in the first equation we have

$$(0.14) \quad U^\dagger [R] X U [R] = \left(I + \frac{i\epsilon_z L_z}{\hbar} \right) X \left(I - \frac{i\epsilon_z L_z}{\hbar} \right)$$

$$(0.15) \quad = X + \frac{i\epsilon_z}{\hbar} (L_z X - X L_z)$$

$$(0.16) \quad = X - Y \epsilon_z$$

Equating the last two lines, we get

$$(0.17) \quad [X, L_z] = -i\hbar Y$$

Similarly, for the other three equations we get

$$(0.18) \quad [Y, L_z] = i\hbar X$$

$$(0.19) \quad [P_x, L_z] = -i\hbar P_y$$

$$(0.20) \quad [P_y, L_z] = i\hbar P_x$$

We can use these commutation relations to derive the form of L_z by using the commutation relations for coordinates and momenta:

$$(0.21) \quad [X, P_x] = [Y, P_y] = i\hbar$$

with all other commutators involving X, Y, P_x and P_y being zero. Starting with 0.17, we see that

$$(0.22) \quad [X, L_z] = -[X, P_x] Y$$

We can therefore deduce that

$$(0.23) \quad L_z = -P_x Y + f(X, Y, P_y)$$

where f is some unknown function. We must include f since the commutators of X with X, Y and P_y are all zero, so adding on f still satisfies 0.17.

(You can think of it as similar to adding on the constant in an indefinite integral.)

Now from 0.18, we have

$$(0.24) \quad [Y, L_z] = [Y, P_y]X$$

so combining this with 0.23 we have

$$(0.25) \quad L_z = -P_x Y + P_y X + g(X, Y)$$

The undetermined function is now a function only of X and Y , since the dependence of L_z on P_x and P_y has been determined uniquely by the commutators 0.17 and 0.18.

From 0.19 we have

$$(0.26) \quad [P_x, L_z] = [P_x, X]P_y$$

We can see that this is satisfied already by 0.25, except that we now know that the function g cannot depend on X , since then $[P_x, g] \neq 0$. Thus we have narrowed down L_z to

$$(0.27) \quad L_z = -P_x Y + P_y X + h(Y)$$

Finally, from 0.20 we have

$$(0.28) \quad [P_y, L_z] = -[P_y, Y]P_x$$

This is satisfied by 0.27 if we take $h = 0$ (well, technically, we could take h to be some constant, but we might as well take the constant to be zero), giving us the final form for L_z :

$$(0.29) \quad L_z = -P_x Y + P_y X$$

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