ROTATIONAL TRANSFORMATIONS USING PASSIVE TRANSFORMATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.2.2.

We can also derive the generator of rotations L_z by considering passive transformations of the position and momentum operators, in a way similar to that used for deriving the generator of translations. In a passive transformation, the operators are modified while the state vectors remain the same. For an infinitesimal rotation $\varepsilon_z\hat{\mathbf{z}}$ about the z axis in two dimensions, the unitary operator has the form

$$U\left[R\left(\varepsilon_z\hat{\mathbf{z}}\right)\right]=I-\frac{i\varepsilon_z L_z}{\hbar} \tag{1}$$

For a finite rotation by $\phi_0\hat{\mathbf{z}}$ the transformations are given by

$$\langle X\rangle_R = \langle X\rangle\cos\phi_0-\langle Y\rangle\sin\phi_0 \tag{2}$$
$$\langle Y\rangle_R = \langle X\rangle\sin\phi_0+\langle Y\rangle\cos\phi_0 \tag{3}$$
$$\langle P_x\rangle_R = \langle P_x\rangle\cos\phi_0-\left\langle P_y\right\rangle\sin\phi_0 \tag{4}$$
$$\left\langle P_y\right\rangle_R = \langle P_x\rangle\sin\phi_0+\left\langle P_y\right\rangle\cos\phi_0 \tag{5}$$

For the infinitesimal transformation, $\phi_0=\varepsilon_z$ and these equations reduce to

$$\langle X\rangle_R = \langle X\rangle-\langle Y\rangle\varepsilon_z \tag{6}$$
$$\langle Y\rangle_R = \langle X\rangle\varepsilon_z+\langle Y\rangle \tag{7}$$
$$\langle P_x\rangle_R = \langle P_x\rangle-\left\langle P_y\right\rangle\varepsilon_z \tag{8}$$
$$\left\langle P_y\right\rangle_R = \langle P_x\rangle\varepsilon_z+\left\langle P_y\right\rangle \tag{9}$$

In the passive transformation scheme, we move the transformation to the operators to get

$$U^{\dagger}[R]XU[R] = X - Y\varepsilon_z \tag{10}$$
$$U^{\dagger}[R]YU[R] = X\varepsilon_z + Y \tag{11}$$
$$U^{\dagger}[R]P_xU[R] = P_x - P_y\varepsilon_z \tag{12}$$
$$U^{\dagger}[R]P_yU[R] = P_x\varepsilon_z + P_y \tag{13}$$

Substituting 1 into these equations gives us the commutation relations satisfied by L_z. For example, in the first equation we have

$$U^{\dagger}[R]XU[R] = \left(I + \frac{i\varepsilon_z L_z}{\hbar}\right) X \left(I - \frac{i\varepsilon_z L_z}{\hbar}\right) \tag{14}$$
$$= X + \frac{i\varepsilon_z}{\hbar}\left(L_z X - X L_z\right) \tag{15}$$
$$= X - Y\varepsilon_z \tag{16}$$

Equating the last two lines, we get

$$[X, L_z] = -i\hbar Y \tag{17}$$

Similarly, for the other three equations we get

$$[Y, L_z] = i\hbar X \tag{18}$$
$$[P_x, L_z] = -i\hbar P_y \tag{19}$$
$$[P_y, L_z] = i\hbar P_x \tag{20}$$

We can use these commutation relations to derive the form of L_z by using the commutation relations for coordinates and momenta:

$$[X, P_x] = [Y, P_y] = i\hbar \tag{21}$$

with all other commutators involving X, Y, P_x and P_y being zero. Starting with 17, we see that

$$[X, L_z] = -[X, P_x]\, Y \tag{22}$$

We can therefore deduce that

$$L_z = -P_x Y + f\left(X, Y, P_y\right) \tag{23}$$

where f is some unknown function. We must include f since the commutators of X with X, Y and P_y are all zero, so adding on f still satisfies

17. (You can think of it as similar to adding on the constant in an indefinite integral.)

Now from 18, we have

$$[Y, L_z] = [Y, P_y] X \tag{24}$$

so combining this with 23 we have

$$L_z = -P_x Y + P_y X + g(X, Y) \tag{25}$$

The undetermined function is now a function only of X and Y, since the dependence of L_z on P_x and P_y has been determined uniquely by the commutators 17 and 18.

From 19 we have

$$[P_x, L_z] = [P_x, X] P_y \tag{26}$$

We can see that this is satisfied already by 25, except that we now know that the function g cannot depend on X, since then $[P_x, g] \neq 0$. Thus we have narrowed down L_z to

$$L_z = -P_x Y + P_y X + h(Y) \tag{27}$$

Finally, from 20 we have

$$[P_y, L_z] = -[P_y, Y] P_x \tag{28}$$

This is satisfied by 27 if we take $h = 0$ (well, technically, we could take h to be some constant, but we might as well take the constant to be zero), giving us the final form for L_z:

$$L_z = -P_x Y + P_y X \tag{29}$$

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