ROTATIONS THROUGH A FINITE ANGLE; USE OF POLAR COORDINATES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.2.3.

The angluar momentum operator L_z is the generator of rotations in the xy plane. We did the derivation for infinitesimal rotations, but we can generalize this to finite rotations in a similar manner to that used for translations. The unitary transformation for an infinitesimal rotation is

$$U\left[R\left(\varepsilon_z\hat{\mathbf{z}}\right)\right]=I-\frac{i\varepsilon_z L_z}{\hbar} \tag{1}$$

For rotation through a finite angle ϕ_0, we divide up the angle into N small angles, so $\varepsilon_z=\phi_0/N$. Rotation through the full angle ϕ_0 is then given by

$$U\left[R\left(\phi_0\hat{\mathbf{z}}\right)\right]=\lim_{N\to\infty}\left(I-\frac{i\phi_0 L_z}{N\hbar}\right)^N=e^{-i\phi_0 L_z/\hbar} \tag{2}$$

The limit follows because the only non-trivial operator involved is L_z, so no commutation problems arise.

In rectangular coordinates, L_z has the relatively non-obvious form

$$\begin{aligned} L_z &= XP_y-YP_x && (3)\\ &= -i\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right) && (4)\end{aligned}$$

so it's not immediately clear that 2 does in fact lead to the desired rotation. Trying to calculate the exponential with L_z expressed this way is not easy, given that the two terms $x\frac{\partial}{\partial y}$ and $y\frac{\partial}{\partial x}$ don't commute.

It turns out that L_z has a much simpler form in polar coordinates, and there are two ways of converting it to polar form. First, we recall the transformation equations.

$$\begin{aligned} x &= \rho\cos\phi && (5)\\ y &= \rho\sin\phi && (6)\\ \rho &= \sqrt{x^2+y^2} && (7)\\ \phi &= \tan^{-1}\frac{y}{x} && (8) \end{aligned}$$

From the chain rule, we can convert the derivatives:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}+\frac{\partial\cos\phi}{\partial x}\frac{\partial}{\partial(\cos\phi)} && (9)\\ &= \frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}-\sin\phi\frac{\partial\phi}{\partial x}\frac{\partial}{(-\sin\phi)\partial\phi} && (10)\\ &= \frac{x}{\rho}\frac{\partial}{\partial\rho}-\sin\phi\frac{-y/x^2}{1+y^2/x^2}\left(\frac{-1}{\sin\phi}\right)\frac{\partial}{\partial\phi} && (11)\\ &= \frac{x}{\rho}\frac{\partial}{\partial\rho}-\frac{y}{\rho^2}\frac{\partial}{\partial\phi} && (12) \end{aligned}$$

Using similar methods, we get for the other derivative

$$\begin{aligned} \frac{\partial}{\partial y} &= \frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho}+\frac{\partial\sin\phi}{\partial x}\frac{\partial}{\partial(\sin\phi)} && (13)\\ &= \frac{y}{\rho}\frac{\partial}{\partial\rho}+\frac{x}{\rho^2}\frac{\partial}{\partial\phi} && (14) \end{aligned}$$

Plugging these into 4 we have

$$\begin{aligned} L_z &= -i\hbar\left[x\left(\frac{y}{\rho}\frac{\partial}{\partial\rho}+\frac{x}{\rho^2}\frac{\partial}{\partial\phi}\right)-y\left(\frac{x}{\rho}\frac{\partial}{\partial\rho}-\frac{y}{\rho^2}\frac{\partial}{\partial\phi}\right)\right] && (15)\\ &= -i\hbar\frac{x^2+y^2}{\rho^2}\frac{\partial}{\partial\phi} && (16)\\ &= -i\hbar\frac{\partial}{\partial\phi} && (17) \end{aligned}$$

Another method of converting L_z to polar coordinates is to consider the effect of $U[R]$ for an infinitesimal rotation ε_z on a state vector expressed in polar coordinates $\psi(\rho,\phi)$. Shankar states that

$$\langle\rho,\phi\,|U[R]|\,\psi(\rho,\phi)\rangle = \psi(\rho,\phi-\varepsilon_z) \qquad (18)$$

If you don't believe this, it can be shown using a method similar to that for the one-dimensional translation. In this case, we're dealing with position eigenkets in polar coordinates, so we have

$$U[R]|\rho,\phi\rangle = |\rho,\phi+\varepsilon_z\rangle \tag{19}$$

Applying this, we get

$$\begin{aligned}
\left|\psi_{\varepsilon_z}\right\rangle &= U[R]|\psi\rangle &(20)\\
&= U[R]\int_0^{2\pi}\int_0^{\infty}|\rho,\phi\rangle\langle\rho,\phi|\psi\rangle\,\rho d\rho\, d\phi &(21)\\
&= \int_0^{2\pi}\int_0^{\infty}|\rho,\phi+\varepsilon_z\rangle\langle\rho,\phi|\psi\rangle\,\rho d\rho\, d\phi &(22)\\
&= \int_0^{2\pi}\int_0^{\infty}\left|\rho',\phi'\right\rangle\left\langle\rho',\phi'-\varepsilon_z|\psi\right\rangle\rho' d\rho'\, d\phi' &(23)
\end{aligned}$$

where in the last line, we used the substitution $\phi' = \phi+\varepsilon_z$. (The substitution $\rho' = \rho$ is used just to give the radial variable a different name in the integrand.) We can use the same limits of integration for ϕ and ϕ', since we just need to ensure that the integral covers the total range of angles. It then follows that

$$\begin{aligned}
\left\langle\rho,\phi\left|\psi_{\varepsilon_z}\right.\right\rangle &= \int_0^{2\pi}\int_0^{\infty}\left\langle\rho,\phi\left|\rho',\phi'\right.\right\rangle\left\langle\rho',\phi'-\varepsilon_z|\psi\right\rangle\rho' d\rho'\, d\phi' &(24)\\
&= \int_0^{2\pi}\int_0^{\infty}\delta\left(\rho-\rho'\right)\delta\left(\phi-\phi'\right)\left\langle\rho',\phi'-\varepsilon_z|\psi\right\rangle\rho' d\rho'\, d\phi' &(25)\\
&= \psi(\rho,\phi-\varepsilon_z) &(26)
\end{aligned}$$

Combining this with 1 we have

$$\left\langle\rho,\phi\left|I-\frac{i\varepsilon_z L_z}{\hbar}\right|\psi\right\rangle = \psi(\rho,\phi-\varepsilon_z) \tag{27}$$

Expanding the RHS to order ε_z we have

$$\left\langle\rho,\phi\left|I-\frac{i\varepsilon_z L_z}{\hbar}\right|\psi\right\rangle = \psi(\rho,\phi)-\varepsilon_z\frac{\partial\psi}{\partial\phi} \tag{28}$$

from which 17 follows again.

Once we have L_z in this form, the exponential form of a finite rotation is easier to interpret, for we have, from 2

$$e^{-i\phi_0 L_z/\hbar} = \exp\left[-\phi_0\frac{\partial}{\partial\phi}\right] \tag{29}$$

$$= 1 - \phi_0\frac{\partial}{\partial\phi} + \frac{\phi_0^2}{2!}\frac{\partial^2}{\partial\phi^2} + \ldots \tag{30}$$

Applying this to a state function $\psi(\rho,\phi)$, we see that we get the Taylor series for $\psi(\rho,\phi-\phi_0)$, so the exponential does indeed represent a rotation through a finite angle.

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