

## ROTATIONS THROUGH A FINITE ANGLE; USE OF POLAR COORDINATES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.2.3.

The angular momentum operator  $L_z$  is the generator of rotations in the  $xy$  plane. We did the derivation for infinitesimal rotations, but we can generalize this to finite rotations in a similar manner to that used for translations. The unitary transformation for an infinitesimal rotation is

$$(0.1) \quad U [R(\epsilon_z \hat{z})] = I - \frac{i\epsilon_z L_z}{\hbar}$$

For rotation through a finite angle  $\phi_0$ , we divide up the angle into  $N$  small angles, so  $\epsilon_z = \phi_0/N$ . Rotation through the full angle  $\phi_0$  is then given by

$$(0.2) \quad U [R(\phi_0 \hat{z})] = \lim_{N \rightarrow \infty} \left( I - \frac{i\phi_0 L_z}{N\hbar} \right)^N = e^{-i\phi_0 L_z/\hbar}$$

The limit follows because the only non-trivial operator involved is  $L_z$ , so no commutation problems arise.

In rectangular coordinates,  $L_z$  has the relatively non-obvious form

$$(0.3) \quad L_z = XP_y - YP_x$$

$$(0.4) \quad = -i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

so it's not immediately clear that 0.2 does in fact lead to the desired rotation. Trying to calculate the exponential with  $L_z$  expressed this way is not easy, given that the two terms  $x \frac{\partial}{\partial y}$  and  $y \frac{\partial}{\partial x}$  don't commute.

It turns out that  $L_z$  has a much simpler form in polar coordinates, and there are two ways of converting it to polar form. First, we recall the transformation equations.

$$(0.5) \quad x = \rho \cos \phi$$

$$(0.6) \quad y = \rho \sin \phi$$

$$(0.7) \quad \rho = \sqrt{x^2 + y^2}$$

$$(0.8) \quad \phi = \tan^{-1} \frac{y}{x}$$

From the chain rule, we can convert the derivatives:

$$(0.9) \quad \frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x} \frac{\partial}{\partial \rho} + \frac{\partial \cos \phi}{\partial x} \frac{\partial}{\partial (\cos \phi)}$$

$$(0.10) \quad = \frac{\partial \rho}{\partial x} \frac{\partial}{\partial \rho} - \sin \phi \frac{\partial \phi}{\partial x} \frac{\partial}{\partial (-\sin \phi)}$$

$$(0.11) \quad = \frac{x}{\rho} \frac{\partial}{\partial \rho} - \sin \phi \frac{-y/x^2}{1 + y^2/x^2} \left( \frac{-1}{\sin \phi} \right) \frac{\partial}{\partial \phi}$$

$$(0.12) \quad = \frac{x}{\rho} \frac{\partial}{\partial \rho} - \frac{y}{\rho^2} \frac{\partial}{\partial \phi}$$

Using similar methods, we get for the other derivative

$$(0.13) \quad \frac{\partial}{\partial y} = \frac{\partial \rho}{\partial y} \frac{\partial}{\partial \rho} + \frac{\partial \sin \phi}{\partial y} \frac{\partial}{\partial (\sin \phi)}$$

$$(0.14) \quad = \frac{y}{\rho} \frac{\partial}{\partial \rho} + \frac{x}{\rho^2} \frac{\partial}{\partial \phi}$$

Plugging these into 0.4 we have

$$(0.15) \quad L_z = -i\hbar \left[ x \left( \frac{y}{\rho} \frac{\partial}{\partial \rho} + \frac{x}{\rho^2} \frac{\partial}{\partial \phi} \right) - y \left( \frac{x}{\rho} \frac{\partial}{\partial \rho} - \frac{y}{\rho^2} \frac{\partial}{\partial \phi} \right) \right]$$

$$(0.16) \quad = -i\hbar \frac{x^2 + y^2}{\rho^2} \frac{\partial}{\partial \phi}$$

$$(0.17) \quad = -i\hbar \frac{\partial}{\partial \phi}$$

Another method of converting  $L_z$  to polar coordinates is to consider the effect of  $U[R]$  for an infinitesimal rotation  $\epsilon_z$  on a state vector expressed in polar coordinates  $\psi(\rho, \phi)$ . Shankar states that

$$(0.18) \quad \langle \rho, \phi | U[R] | \psi(\rho, \phi) \rangle = \psi(\rho, \phi - \epsilon_z)$$

If you don't believe this, it can be shown using a method similar to that for the one-dimensional translation. In this case, we're dealing with position eigenkets in polar coordinates, so we have

$$(0.19) \quad U[R]|\rho, \phi\rangle = |\rho, \phi + \varepsilon_z\rangle$$

Applying this, we get

$$(0.20) \quad |\psi_{\varepsilon_z}\rangle = U[R]|\psi\rangle$$

$$(0.21) \quad = U[R] \int_0^{2\pi} \int_0^\infty |\rho, \phi\rangle \langle \rho, \phi | \psi \rangle \rho d\rho d\phi$$

$$(0.22) \quad = \int_0^{2\pi} \int_0^\infty |\rho, \phi + \varepsilon_z\rangle \langle \rho, \phi | \psi \rangle \rho d\rho d\phi$$

$$(0.23) \quad = \int_0^{2\pi} \int_0^\infty |\rho', \phi'\rangle \langle \rho', \phi' - \varepsilon_z | \psi \rangle \rho' d\rho' d\phi'$$

where in the last line, we used the substitution  $\phi' = \phi + \varepsilon_z$ . (The substitution  $\rho' = \rho$  is used just to give the radial variable a different name in the integrand.) We can use the same limits of integration for  $\phi$  and  $\phi'$ , since we just need to ensure that the integral covers the total range of angles. It then follows that

$$(0.24)$$

$$\langle \rho, \phi | \psi_{\varepsilon_z} \rangle = \int_0^{2\pi} \int_0^\infty \langle \rho, \phi | \rho', \phi' \rangle \langle \rho', \phi' - \varepsilon_z | \psi \rangle \rho' d\rho' d\phi'$$

$$(0.25) \quad = \int_0^{2\pi} \int_0^\infty \delta(\rho - \rho') \delta(\phi - \phi') \langle \rho', \phi' - \varepsilon_z | \psi \rangle \rho' d\rho' d\phi'$$

$$(0.26) \quad = \psi(\rho, \phi - \varepsilon_z)$$

Combining this with 0.1 we have

$$(0.27) \quad \left\langle \rho, \phi \left| I - \frac{i\varepsilon_z L_z}{\hbar} \right| \psi \right\rangle = \psi(\rho, \phi - \varepsilon_z)$$

Expanding the RHS to order  $\varepsilon_z$  we have

$$(0.28) \quad \left\langle \rho, \phi \left| I - \frac{i\varepsilon_z L_z}{\hbar} \right| \psi \right\rangle = \psi(\rho, \phi) - \varepsilon_z \frac{\partial \psi}{\partial \phi}$$

from which 0.17 follows again.

Once we have  $L_z$  in this form, the exponential form of a finite rotation is easier to interpret, for we have, from 0.2

$$(0.29) \quad e^{-i\phi_0 L_z/\hbar} = \exp \left[ -\phi_0 \frac{\partial}{\partial \phi} \right]$$

$$(0.30) \quad = 1 - \phi_0 \frac{\partial}{\partial \phi} + \frac{\phi_0^2}{2!} \frac{\partial^2}{\partial \phi^2} + \dots$$

Applying this to a state function  $\psi(\rho, \phi)$ , we see that we get the Taylor series for  $\psi(\rho, \phi - \phi_0)$ , so the exponential does indeed represent a rotation through a finite angle.

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