COMBINING TRANSLATIONS AND ROTATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.2.4.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

When it comes to symmetries in quantum mechanics, we've looked at translations and rotations in two dimensions, and found that the generators are the momenta P_x and P_y for translations, and the angular momentum L_z for rotations.

From the fact that L_z does not commute with either momentum or position operators, you might guess that if we performed some sequence of translations and rotations on a system that the order in which these operations are done matters. In fact, you can see this by considering simple two-dimensional geometry, without reference to quantum mechanics. Consider the x and y axes on a sheet of graph paper. First, translate these axes by adding the vector $\mathbf{r}$ to all points, so that the new origin of coordinates lies at position $\mathbf{r}$ as referenced in the original coordinates. Next, do a rotation *about the original origin* by some angle ϕ. This will move the new origin around the original z axis. Now, do the inverse of the original translation by adding $-\mathbf{r}$ to all points. Finally, do the inverse of the rotation by rotating the system by $-\phi$ around the *original* z axis. You'll find that the xy axes that have undergone this sequence of transformations does not coincide with the original xy axes. However, if you did the same set of four transformations in the order: translate by $\mathbf{r}$, translate by $-\mathbf{r}$, rotate by ϕ, rotate by $-\phi$, the transformed axes *would* coincide with the original axes.

To see how this works in quantum mechanics, we can again consider infinitesimal translations and rotations. If we start with a point at location $[x,y]$ and apply the four transformations described above, but now for an infinitesimal translation $\boldsymbol{\varepsilon}=\varepsilon_x\hat{\mathbf{x}}+\varepsilon_y\hat{\mathbf{y}}$ and rotation $\varepsilon_z\hat{\mathbf{z}}$, then the successive transformations work as follows. In each case, we'll retain terms up to order $\varepsilon_x\varepsilon_z$ and $\varepsilon_y\varepsilon_z$ but discard terms of order ε_x^2, ε_y^2, ε_z^2 and higher. [I'm not quite sure of the rationale that allows us to do this, apart from the fact that it gives the right answer.]

$$\begin{bmatrix} x \\ y \end{bmatrix} \underset{T(\varepsilon)}{\longrightarrow} \begin{bmatrix} x+\varepsilon_x \\ y+\varepsilon_y \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} x+\varepsilon_x \\ y+\varepsilon_y \end{bmatrix} \underset{R(\varepsilon_z\hat{\mathbf{z}})}{\longrightarrow} \begin{bmatrix} x+\varepsilon_x-(y+\varepsilon_y)\,\varepsilon_z \\ y+\varepsilon_y+(x+\varepsilon_x)\,\varepsilon_z \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} x+\varepsilon_x-(y+\varepsilon_y)\,\varepsilon_z \\ y+\varepsilon_y+(x+\varepsilon_x)\,\varepsilon_z \end{bmatrix} \underset{T(-\varepsilon)}{\longrightarrow} \begin{bmatrix} x+\varepsilon_x-(y+\varepsilon_y)\,\varepsilon_z-\varepsilon_x \\ y+\varepsilon_y+(x+\varepsilon_x)\,\varepsilon_z-\varepsilon_y \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} x-(y+\varepsilon_y)\,\varepsilon_z \\ y+(x+\varepsilon_x)\,\varepsilon_z \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} x-(y+\varepsilon_y)\,\varepsilon_z \\ y+(x+\varepsilon_x)\,\varepsilon_z \end{bmatrix} \underset{R(-\varepsilon_z\hat{\mathbf{z}})}{\longrightarrow} \begin{bmatrix} x-(y+\varepsilon_y)\,\varepsilon_z+[y+(x+\varepsilon_x)\,\varepsilon_z]\,\varepsilon_z \\ y+(x+\varepsilon_x)\,\varepsilon_z-[x-(y+\varepsilon_y)\,\varepsilon_z]\,\varepsilon_z \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} x-\varepsilon_y\varepsilon_z \\ y+\varepsilon_x\varepsilon_z \end{bmatrix} \tag{6}$$

Thus, to this order in the infinitesimals, the combination of translation-rotation-translation-rotation is equivalent to a single translation by a distance $[-\varepsilon_y\varepsilon_z, \varepsilon_x\varepsilon_z]$. We can write this in terms of the unitary quantum operators for translations and rotations as

$$U\left[R\left(-\varepsilon_z\hat{\mathbf{z}}\right)\right]T\left(-\varepsilon\right)U\left[R\left(\varepsilon_z\hat{\mathbf{z}}\right)\right]T\left(\varepsilon\right) = T\left(-\varepsilon_y\varepsilon_z\hat{\mathbf{x}}+\varepsilon_x\varepsilon_z\hat{\mathbf{y}}\right) \tag{7}$$

Using the forms of these operators for infinitesimal transformations, we can expand both sides to give

$$\left(I+\frac{i\varepsilon_z}{\hbar}L_z\right)\left[I+\frac{i}{\hbar}\left(\varepsilon_xP_x+\varepsilon_yP_y\right)\right]\times \tag{8}$$

$$\left(I-\frac{i\varepsilon_z}{\hbar}L_z\right)\left[I-\frac{i}{\hbar}\left(\varepsilon_xP_x+\varepsilon_yP_y\right)\right] = I-\frac{i}{\hbar}\left(-\varepsilon_y\varepsilon_zP_x+\varepsilon_x\varepsilon_zP_y\right) \tag{9}$$

Since the infinitesimal displacements are arbitrary, this equation can be valid only if the coefficients of each combination of $\varepsilon_x, \varepsilon_y$ and ε_z are equal on both sides. As above, we'll discard any terms of order ε_x^2, ε_y^2, ε_z^2 and higher. The algebra is straightforward although a bit tedious, so I'll just give a couple of examples here.

The coefficient of ε_z on its own is, on the LHS

$$\frac{i\varepsilon_z}{\hbar}L_z-\frac{i\varepsilon_z}{\hbar}L_z = 0 \tag{10}$$

On the RHS, there is no term in ε_z, so we get 0 on the RHS. In this case, we see the equation is consistent.

For the $\varepsilon_x\varepsilon_z$ term, we get on the LHS:

$$\varepsilon_x\varepsilon_z\frac{i^2}{\hbar^2}\left(L_zP_x-L_zP_x-P_xL_z+L_zP_x\right)=-\varepsilon_x\varepsilon_z\frac{i^2}{\hbar^2}\left[P_x,L_z\right] \tag{11}$$

On the RHS, the term is

$$-\frac{i}{\hbar}\varepsilon_x\varepsilon_zP_y \tag{12}$$

Thus the condition here becomes

$$[P_x,L_z]=-i\hbar P_y \tag{13}$$

which agrees with the commutation relation we found earlier. By considering the coefficient of $\varepsilon_y\varepsilon_z$, we arrive at the other condition, which is

$$[P_y,L_z]=i\hbar P_x \tag{14}$$

The result of this calculation doesn't tell us anything new about the translation or rotation operators, but it does show that the condition 7 is consistent with what we already know about the commutators of position, momentum and angular momentum.

As Shankar points out, we might think that we need to verify the conditions for an infinite number of combinations of rotations and translations, since each such combination gives rise to a different overall transformation. He says that it has actually been shown that the example above is sufficient to guarantee that all such combinations do in fact give valid results, although he doesn't give the details. We are, however, given the exercise of verifying this claim for one special case, which we'll consider now.

In this example, we'll consider the same four transformations, in the same order, as above except that we'll take the translation to be entirely in the x direction so that $\varepsilon_y=0$. This time, we'll retain terms up to $\varepsilon_x\varepsilon_z^2$ and see what we get. We start by repeating the calculations in 1 through 6. However, because we're saving higher order terms, we need to represent the infinitesimal rotations by

$$R\left(\varepsilon_z\hat{\mathbf{z}}\right)=\begin{bmatrix}1-\frac{\varepsilon_z^2}{2} & -\varepsilon_z\\ \varepsilon_z & 1-\frac{\varepsilon_z^2}{2}\end{bmatrix} \tag{15}$$

That is, we're approximating $\cos\varepsilon_z$ by the first two terms in its expansion. Using this, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} \underset{T(\varepsilon)}{\longrightarrow} \begin{bmatrix} x+\varepsilon_x \\ y \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} x+\varepsilon_x \\ y \end{bmatrix} \underset{R(\varepsilon_z\hat{\mathbf{z}})}{\longrightarrow} \begin{bmatrix} (x+\varepsilon_x)\left(1-\frac{\varepsilon_z^2}{2}\right) - y\varepsilon_z \\ y\left(1-\frac{\varepsilon_z^2}{2}\right) + (x+\varepsilon_x)\varepsilon_z \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} x+\varepsilon_x - y\varepsilon_z \\ y+(x+\varepsilon_x)\varepsilon_z \end{bmatrix} \underset{T(-\varepsilon)}{\longrightarrow} \begin{bmatrix} (x+\varepsilon_x)\left(1-\frac{\varepsilon_z^2}{2}\right) - y\varepsilon_z - \varepsilon_x \\ y\left(1-\frac{\varepsilon_z^2}{2}\right) + (x+\varepsilon_x)\varepsilon_z \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} x\left(1-\frac{\varepsilon_z^2}{2}\right) - y\varepsilon_z - \frac{1}{2}\varepsilon_x\varepsilon_z^2 \\ y\left(1-\frac{\varepsilon_z^2}{2}\right) + \varepsilon_z x + \varepsilon_x\varepsilon_z \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} x - y\varepsilon_z \\ y+(x+\varepsilon_x)\varepsilon_z \end{bmatrix} \underset{R(-\varepsilon_z\hat{\mathbf{z}})}{\longrightarrow} \begin{bmatrix} \left[x\left(1-\frac{\varepsilon_z^2}{2}\right) - y\varepsilon_z - \frac{1}{2}\varepsilon_x\varepsilon_z^2\right]\left(1-\frac{\varepsilon_z^2}{2}\right) + \left[y\left(1-\frac{\varepsilon_z^2}{2}\right) + \varepsilon_z x + \varepsilon_x\varepsilon_z\right]\varepsilon_z \\ \left[y\left(1-\frac{\varepsilon_z^2}{2}\right) + \varepsilon_z x + \varepsilon_x\varepsilon_z\right]\left(1-\frac{\varepsilon_z^2}{2}\right) - \left[x\left(1-\frac{\varepsilon_z^2}{2}\right) - y\varepsilon_z - \frac{1}{2}\varepsilon_x\varepsilon_z^2\right]\varepsilon_z \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} x\left(1+\frac{\varepsilon_z^4}{4}\right) + \frac{1}{2}\varepsilon_x\varepsilon_z^2 + \frac{1}{4}\varepsilon_x\varepsilon_z^4 \\ y\left(1+\frac{\varepsilon_z^4}{4}\right) + \varepsilon_x\varepsilon_z \end{bmatrix} \tag{21}$$

To get the last line, I used Maple to do the algebra in multiplying out the terms. At this point, we can neglect the terms in ε_z^4, leaving us with the overall transformation:

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x+\frac{1}{2}\varepsilon_x\varepsilon_z^2 \\ y+\varepsilon_x\varepsilon_z \end{bmatrix} \tag{22}$$

This is equivalent to a translation by $\varepsilon = \frac{1}{2}\varepsilon_x\varepsilon_z^2\hat{\mathbf{x}} + \varepsilon_x\varepsilon_z\hat{\mathbf{y}}$, so by analogy with 7, we have the condition

$$U[R(-\varepsilon_z\hat{\mathbf{z}})]\,T(-\varepsilon)\,U[R(\varepsilon_z\hat{\mathbf{z}})]\,T(\varepsilon) = T\left(\frac{1}{2}\varepsilon_x\varepsilon_z^2\hat{\mathbf{x}} + \varepsilon_x\varepsilon_z\hat{\mathbf{y}}\right) \tag{23}$$

To expand the operators on the LHS and retain terms up to $\varepsilon_x\varepsilon_z^2$, we need to expand the rotation operators up to order ε_z^2. Treating the rotation operator as an exponential, this expansion is

$$R(\varepsilon_z\hat{\mathbf{z}}) = I - \frac{i\varepsilon_z}{\hbar}L_z + \frac{i^2\varepsilon_z^2}{2\hbar^2}L_z^2 + \ldots \tag{24}$$

Using this approximation gives us

$$\left(I+\frac{i\varepsilon_z}{\hbar}L_z+\frac{i^2\varepsilon_z^2}{2\hbar^2}L_z^2\right)\left[I+\frac{i}{\hbar}\varepsilon_x P_x\right]\left(I-\frac{i\varepsilon_z}{\hbar}L_z+\frac{i^2\varepsilon_z^2}{2\hbar^2}L_z^2\right)\left[I-\frac{i}{\hbar}\varepsilon_x P_x\right] = I-\frac{i}{\hbar}\left(\frac{1}{2}\varepsilon_x\varepsilon_z^2 P_x+\varepsilon_x\varepsilon_z P_y\right) \tag{25}$$

By equating the coefficients of $\varepsilon_x\varepsilon_z$ we regain 13, so that condition checks out.

Extracting the coefficient of $\varepsilon_x\varepsilon_z^2$ on the LHS gives

$$\frac{i^3}{\hbar^3}\varepsilon_x\varepsilon_z^2\left(-L_zP_xL_z+\frac{L_z^2P_x}{2}-\frac{L_z^2P_x}{2}+\frac{P_xL_z^2}{2}-\frac{L_z^2P_x}{2}+L_z^2P_x\right) = \frac{i^3}{\hbar^3}\varepsilon_x\varepsilon_z^2\left(-L_zP_xL_z+\frac{L_z^2P_x}{2}+\frac{P_xL_z^2}{2}\right) \tag{26}$$

Matching this to the $\varepsilon_x\varepsilon_z^2$ term on the RHS of 25, we get the condition specified in Shankar's problem:

$$-2L_zP_xL_z+L_z^2P_x+P_xL_z^2=\hbar^2P_x \tag{27}$$

We can show that this condition reduces to the already-known commutators by using the identity

$$\begin{aligned} [\Lambda,[\Lambda,\Omega]] &= \Lambda(\Lambda\Omega-\Omega\Lambda)-(\Lambda\Omega-\Omega\Lambda)\Lambda && (28)\\ &= -2\Lambda\Omega\Lambda+\Lambda^2\Omega+\Omega\Lambda^2 && (29) \end{aligned}$$

Applying this to 27 we have

$$\begin{aligned} -2L_zP_xL_z+L_z^2P_x+P_xL_z^2 &= [L_z,[L_z,P_x]] && (30)\\ &= i\hbar[L_z,P_y] && (31)\\ &= i\hbar(-i\hbar P_x) && (32)\\ &= \hbar^2P_x && (33) \end{aligned}$$

Thus the more complicated condition 27 actually reduces to existing commutators.