EIGENVALUES OF TWO-DIMENSIONAL ANGULAR MOMENTUM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.3.1.

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The angular momentum operator L_z for rotations in two dimensions has the form, in polar coordinates, of

$$L_z = -i\hbar\frac{\partial}{\partial\phi} \tag{1}$$

To find the eigenvalues and eigenfunctions, we need to solve

$$L_z\left|\ell_z\right\rangle = \ell_z\left|\ell_z\right\rangle \tag{2}$$

where $\left|\ell_z\right\rangle$ is the eigenfunction and ℓ_z is the corresponding eigenvalue. Using polar coordinates, we must solve

$$-i\hbar\frac{\partial}{\partial\phi}\psi_{\ell_z}(\rho,\phi) = \ell_z\psi_{\ell_z}(\rho,\phi) \tag{3}$$

where ρ is the radial coordinate. As the only derivative here is with respect to ϕ, we can solve this using separation of variables by proposing a solution of form

$$\psi_{\ell_z}(\rho,\phi) = R(\rho)\Phi(\phi) \tag{4}$$

Substituting this and cancelling off $R(\rho)$ we get

$$-i\hbar\frac{\partial}{\partial\phi}\Phi(\phi) = \ell_z\Phi(\phi) \tag{5}$$

which has the solution

$$\Phi(\phi) = Ae^{i\ell_z\phi/\hbar} \tag{6}$$

for some constant A, which we can absorb into $R(\rho)$ to give the general solution

$$\psi_{\ell_z}(\rho,\phi) = R(\rho)e^{i\ell_z\phi/\hbar} \tag{7}$$

[This is actually the two-dimensional version of the more general 3-d case, in which the solution involved a radial function multiplied by a spherical harmonic.]

At this stage, the eigenvalue ℓ_z could be any number, real or complex, since they all satisfy 3. However, since L_z is an observable, it must be hermitian, which implies that $L_z^\dagger = L_z$, so that

$$\langle\psi_1|L_z|\psi_2\rangle = \langle\psi_2|L_z|\psi_1\rangle^* \tag{8}$$

In the coordinate basis, we have

$$\int_0^\infty \int_0^{2\pi} \psi_1^* \left(-i\hbar\frac{\partial}{\partial\phi}\right)\psi_2 d\phi\, d\rho = \left[\int_0^\infty \int_0^{2\pi} \psi_2^* \left(-i\hbar\frac{\partial}{\partial\phi}\right)\psi_1 d\phi\, d\rho\right]^* \tag{9}$$

Integrating the LHS by parts, we have

$$\int_0^\infty \int_0^{2\pi} \psi_1^* \left(-i\hbar\frac{\partial}{\partial\phi}\right)\psi_2 d\phi\, d\rho = -i\hbar\int_0^\infty \psi_1^*\psi_2|_0^{2\pi}\, d\rho + i\hbar\int_0^\infty \int_0^{2\pi} \frac{\partial\psi_1^*}{\partial\phi}\psi_2 d\phi\, d\rho \tag{10}$$

The second term on the RHS is seen to be equal to the RHS of 9, so in order for 9 to be true, we must have

$$\int_0^\infty \psi_1^*\psi_2|_0^{2\pi}\, d\rho = 0 \tag{11}$$

Although two different eigenfunctions ψ_1 and ψ_2 are orthogonal and thus would satisfy this condition automatically, the condition must also be true when $\psi_1 = \psi_2$. This gives us the condition that

$$\psi_{\ell_z}(2\pi) = \psi_{\ell_z}(0) \tag{12}$$

That is, the eigenfunctions must be periodic with period 2π. Looking back at 7, we see that this forces the eigenvalues ℓ_z to be integral multiples of $\hbar$:

$$\ell_z = m\hbar \tag{13}$$

$$m = 0, \pm1, \pm2, \ldots \tag{14}$$

Here m is the *magnetic quantum number*, not the mass.

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