EIGENVALUES OF ANGULAR MOMENTUM

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.3.2.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

One consequence of requiring the angular momentum operator L_z to be hermitian is that the eigenvalues must be integral multiples of $\hbar$, so that $\ell_z = m\hbar$ for $m = 0, \pm 1, \pm 2, \ldots$. Shankar proposes another method by which we might try to obtain this restriction on ℓ_z. We start with a superposition of two eigenstates of L_z, so that

$$\psi(\rho,\phi) = A(\rho)\, e^{i\phi\ell_z/\hbar} + B(\rho)\, e^{i\phi\ell_z'/\hbar} \tag{1}$$

$$= e^{i\phi\ell_z'/\hbar}\left[A(\rho)\, e^{i\phi(\ell_z-\ell_z')/\hbar} + B(\rho)\right] \tag{2}$$

where A and B are two unknown functions of the radial coordinate ρ, and ℓ_z and ℓ_z' are two eigenvalues of L_z. If we rotate the system by a complete circle, so that $\phi \to \phi + 2\pi$, the physical state should remain unchanged. This means that

$$|\psi(\rho,\phi+2\pi)| = |\psi(\rho,\phi)| \tag{3}$$

so that $\psi(\rho,\phi+2\pi)$ may differ from $\psi(\rho,\phi)$ by a phase factor. From 2

$$\psi(\rho,\phi+2\pi) = e^{i(\phi+2\pi)\ell_z'/\hbar}\left[A(\rho)\, e^{i(\phi+2\pi)(\ell_z-\ell_z')/\hbar} + B(\rho)\right] \tag{4}$$

The phase factor of $e^{i(\phi+2\pi)\ell_z'/\hbar}$ on the RHS can be anything (provided the exponent is purely imaginary), but the quantity in the square brackets must be numerically the same as the corresponding quantity in 2. This means that

$$\frac{(\phi+2\pi)(\ell_z-\ell_z')}{\hbar} = \frac{\phi(\ell_z-\ell_z')}{\hbar} + 2m\pi \tag{5}$$

where m is an integer. This gives the condition

$$\ell_z - \ell_z' = m\hbar \tag{6}$$

To proceed further, we need to argue that ℓ_z is symmetric about zero, that is, if ℓ_z is an eigenvalue, then so is $-\ell_z$. I'm not sure if Shankar expects us to prove this rigorously, but it seems plausible, since the only difference between $+\ell_z$ and $-\ell_z$ is (classically, anyway) that the direction of rotation is reversed. Given this condition, ℓ_z must be a multiple of $\frac{1}{2}\hbar$, since any other value doesn't satisfy both the conditions of symmetry about zero, and 6. (For example, if we try $\ell_z = \frac{1}{4}\hbar$, then the symmetry requirement means we must also allow $\ell_z = -\frac{1}{4}\hbar$, but this violates 6.) If ℓ_z is an odd multiple of $\frac{1}{2}\hbar$, then we get the sequence $\ldots, -\frac{3}{2}\hbar$,-$\frac{1}{2}\hbar, +\frac{1}{2}\hbar, +\frac{3}{2}\hbar, \ldots$ while if ℓ_z is an even multiple of $\frac{1}{2}\hbar$ we get the sequence $\ldots, -2\hbar, -\hbar, 0, +\hbar, +2\hbar, \ldots$. In reality, only the latter sequence is correct, but we can't show that from this argument.