

ANGULAR MOMENTUM: PROBABILITIES OF EIGENVALUES IN TWO DIMENSIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercises 12.3.3 - 12.3.4.

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We've seen that the eigenfunctions of two-dimensional angular momentum have the form

$$\psi(\rho, \phi) = R(\rho) e^{i\ell_z \phi / \hbar} \quad (1)$$

where ℓ_z (the eigenvalue) is an integral multiple of \hbar and $R(\rho)$ is some function of the radial coordinate ρ which depends on the particular potential function in the hamiltonian. It's more convenient to write the angular function as

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad (2)$$

This set of functions is orthonormal over the interval $\phi \in [0, 2\pi]$, that is

$$\int_0^{2\pi} \Phi_m^*(\phi) \Phi_{m'}(\phi) d\phi = \delta_{mm'} \quad (3)$$

This set of functions forms the angular part of the eigenfunctions of L_z , which in some cases allows us to determine the probabilities of a system being in a particular eigenstate of L_z . Here are a couple of examples.

Example 1. A particle is described by the wave function

$$\psi(\rho, \phi) = A e^{-\rho^2/2\Delta^2} \cos^2 \phi \quad (4)$$

where A is a normalization constant, and Δ is another constant.

We can use the trig identity

$$\cos^2 \phi = \frac{1}{2} (1 + \cos 2\phi) \quad (5)$$

to write this wave function as

$$\psi(\rho, \phi) = \frac{A}{2} e^{-\rho^2/2\Delta^2} [1 + \cos 2\phi] \quad (6)$$

$$= \frac{A}{2} e^{-\rho^2/2\Delta^2} \left(1 + \frac{e^{2i\phi} + e^{-2i\phi}}{2} \right) \quad (7)$$

$$= \frac{A\sqrt{2\pi}}{2} e^{-\rho^2/2\Delta^2} \left(\Phi_0 + \frac{1}{2} (\Phi_2 + \Phi_{-2}) \right) \quad (8)$$

Thus the wave function has the form

$$\psi(\rho, \phi) = c_0 \Phi_0 + c_2 \Phi_2 + c_{-2} \Phi_{-2} \quad (9)$$

where the coefficients c_m can be found by comparison with 8. Since the Φ_m are orthonormal functions, the probability of the particle being in state i is

$$P(\ell_z = m\hbar) = \frac{|c_m|^2}{\sum_j |c_j|^2} \quad (10)$$

We can see from this formula that the factor of $\frac{A\sqrt{2\pi}}{2} e^{-\rho^2/2\Delta^2}$ cancels out of the probability formula, so we have

$$P(\ell_z = 0) = \frac{|c_0|^2}{\sum_j |c_j|^2} \quad (11)$$

$$= \frac{1}{1 + \frac{1}{4} + \frac{1}{4}} \quad (12)$$

$$= \frac{2}{3} \quad (13)$$

$$P(\ell_z = 2\hbar) = \frac{|c_2|^2}{\sum_j |c_j|^2} \quad (14)$$

$$= \frac{\frac{1}{4}}{1 + \frac{1}{4} + \frac{1}{4}} \quad (15)$$

$$= \frac{1}{6} \quad (16)$$

$$P(\ell_z = -2\hbar) = \frac{|c_{-2}|^2}{\sum_j |c_j|^2} \quad (17)$$

$$= \frac{\frac{1}{4}}{1 + \frac{1}{4} + \frac{1}{4}} \quad (18)$$

$$= \frac{1}{6} \quad (19)$$

Example 2. Now we have the wave function

$$\psi(\rho, \phi) = Ae^{-\rho^2/2\Delta^2} \left(\frac{\rho}{\Delta} \cos \phi + \sin \phi \right) \quad (20)$$

Again, we write the trig functions in terms of Φ_m to get

$$\psi(\rho, \phi) = Ae^{-\rho^2/2\Delta^2} \left(\frac{\rho}{\Delta} \frac{e^{i\phi} + e^{-i\phi}}{2} + \frac{e^{i\phi} - e^{-i\phi}}{2i} \right) \quad (21)$$

$$= A\sqrt{2\pi}e^{-\rho^2/2\Delta^2} \left[\left(\frac{\rho}{2\Delta} + \frac{1}{2i} \right) \Phi_1 + \left(\frac{\rho}{2\Delta} - \frac{1}{2i} \right) \Phi_{-1} \right] \quad (22)$$

As above, the factor of $A\sqrt{2\pi}e^{-\rho^2/2\Delta^2}$ cancels out when calculating probabilities, so we have

$$P(\ell_z = \hbar) = \frac{|c_1|^2}{|c_1|^2 + |c_{-1}|^2} \quad (23)$$

$$= \frac{\left| \frac{\rho}{2\Delta} + \frac{1}{2i} \right|^2}{\left| \frac{\rho}{2\Delta} + \frac{1}{2i} \right|^2 + \left| \frac{\rho}{2\Delta} - \frac{1}{2i} \right|^2} \quad (24)$$

$$= \frac{\left(\frac{\rho}{2\Delta} \right)^2 + \frac{1}{4}}{2 \left[\left(\frac{\rho}{2\Delta} \right)^2 + \frac{1}{4} \right]} \quad (25)$$

$$= \frac{1}{2} \quad (26)$$

$$P(\ell_z = -\hbar) = \frac{|c_{-1}|^2}{|c_1|^2 + |c_{-1}|^2} \quad (27)$$

$$= \frac{\left| \frac{\rho}{2\Delta} - \frac{1}{2i} \right|^2}{\left| \frac{\rho}{2\Delta} + \frac{1}{2i} \right|^2 + \left| \frac{\rho}{2\Delta} - \frac{1}{2i} \right|^2} \quad (28)$$

$$= \frac{\left(\frac{\rho}{2\Delta} \right)^2 + \frac{1}{4}}{2 \left[\left(\frac{\rho}{2\Delta} \right)^2 + \frac{1}{4} \right]} \quad (29)$$

$$= \frac{1}{2} \quad (30)$$

Thus in this case, the ρ dependence cancels out when calculating the probabilities, although we can't expect this to be true in general.