ANGULAR MOMENTUM: PROBABILITIES OF EIGENVALUES IN TWO DIMENSIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercises 12.3.3 - 12.3.4.

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We've seen that the eigenfunctions of two-dimensional angular momentum have the form

$$\psi(\rho,\phi)=R(\rho)\,e^{i\ell_z\phi/\hbar} \tag{1}$$

where ℓ_z (the eigenvalue) is an integral multiple of $\hbar$ and $R(\rho)$ is some function of the radial coordinate ρ which depends on the particular potential function in the hamiltonian. It's more convenient to write the angular function as

$$\Phi_m(\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi} \tag{2}$$

This set of functions is orthonormal over the interval $\phi\in[0,2\pi]$, that is

$$\int_0^{2\pi}\Phi_m^*(\phi)\,\Phi_{m'}(\phi)\,d\phi=\delta_{mm'} \tag{3}$$

This set of functions forms the angular part of the eigenfunctions of L_z, which in some cases allows us to determine the probabilities of a system being in a particular eigenstate of L_z. Here are a couple of examples.

Example 1. A particle is described by the wave function

$$\psi(\rho,\phi)=Ae^{-\rho^2/2\Delta^2}\cos^2\phi \tag{4}$$

where A is a normalization constant, and Δ is another constant.

We can use the trig identity

$$\cos^2\phi=\frac{1}{2}(1+\cos2\phi) \tag{5}$$

to write this wave function as

$$\psi(\rho,\phi) = \frac{A}{2}e^{-\rho^2/2\Delta^2}\left[1+\cos 2\phi\right] \tag{6}$$

$$= \frac{A}{2}e^{-\rho^2/2\Delta^2}\left(1+\frac{e^{2i\phi}+e^{-2i\phi}}{2}\right) \tag{7}$$

$$= \frac{A\sqrt{2\pi}}{2}e^{-\rho^2/2\Delta^2}\left(\Phi_0+\frac{1}{2}\left(\Phi_2+\Phi_{-2}\right)\right) \tag{8}$$

Thus the wave function has the form

$$\psi(\rho,\phi) = c_0\Phi_0 + c_2\Phi_2 + c_{-2}\Phi_{-2} \tag{9}$$

where the coefficients c_m can be found by comparison with 8. Since the Φ_m are orthonormal functions, the probability of the particle being in state i is

$$P\left(\ell_z = m\hbar\right) = \frac{\left|c_m\right|^2}{\sum_j \left|c_j\right|^2} \tag{10}$$

We can see from this formula that the factor of $\frac{A\sqrt{2\pi}}{2}e^{-\rho^2/2\Delta^2}$ cancels out of the probability formula, so we have

$$P(\ell_z = 0) = \frac{|c_0|^2}{\sum_j |c_j|^2} \tag{11}$$

$$= \frac{1}{1+\frac{1}{4}+\frac{1}{4}} \tag{12}$$

$$= \frac{2}{3} \tag{13}$$

$$P(\ell_z = 2\hbar) = \frac{|c_2|^2}{\sum_j |c_j|^2} \tag{14}$$

$$= \frac{\frac{1}{4}}{1+\frac{1}{4}+\frac{1}{4}} \tag{15}$$

$$= \frac{1}{6} \tag{16}$$

$$P(\ell_z = -2\hbar) = \frac{|c_{-2}|^2}{\sum_j |c_j|^2} \tag{17}$$

$$= \frac{\frac{1}{4}}{1+\frac{1}{4}+\frac{1}{4}} \tag{18}$$

$$= \frac{1}{6} \tag{19}$$

Example 2. Now we have the wave function

$$\psi(\rho,\phi) = Ae^{-\rho^2/2\Delta^2}\left(\frac{\rho}{\Delta}\cos\phi + \sin\phi\right) \tag{20}$$

Again, we write the trig functions in terms of Φ_m to get

$$\psi(\rho,\phi) = Ae^{-\rho^2/2\Delta^2}\left(\frac{\rho}{\Delta}\frac{e^{i\phi}+e^{-i\phi}}{2} + \frac{e^{i\phi}-e^{-i\phi}}{2i}\right) \tag{21}$$

$$= A\sqrt{2\pi}e^{-\rho^2/2\Delta^2}\left[\left(\frac{\rho}{2\Delta}+\frac{1}{2i}\right)\Phi_1 + \left(\frac{\rho}{2\Delta}-\frac{1}{2i}\right)\Phi_{-1}\right] \tag{22}$$

As above, the factor of $A\sqrt{2\pi}e^{-\rho^2/2\Delta^2}$ cancels out when calculating probabilities, so we have

$$P(\ell_z = \hbar) = \frac{|c_1|^2}{|c_1|^2 + |c_{-1}|^2} \tag{23}$$

$$= \frac{\left|\frac{\rho}{2\Delta} + \frac{1}{2i}\right|^2}{\left|\frac{\rho}{2\Delta} + \frac{1}{2i}\right|^2 + \left|\frac{\rho}{2\Delta} - \frac{1}{2i}\right|^2} \tag{24}$$

$$= \frac{\left(\frac{\rho}{2\Delta}\right)^2 + \frac{1}{4}}{2\left[\left(\frac{\rho}{2\Delta}\right)^2 + \frac{1}{4}\right]} \tag{25}$$

$$= \frac{1}{2} \tag{26}$$

$$P(\ell_z = -\hbar) = \frac{|c_{-1}|^2}{|c_1|^2 + |c_{-1}|^2} \tag{27}$$

$$= \frac{\left|\frac{\rho}{2\Delta} - \frac{1}{2i}\right|^2}{\left|\frac{\rho}{2\Delta} + \frac{1}{2i}\right|^2 + \left|\frac{\rho}{2\Delta} - \frac{1}{2i}\right|^2} \tag{28}$$

$$= \frac{\left(\frac{\rho}{2\Delta}\right)^2 + \frac{1}{4}}{2\left[\left(\frac{\rho}{2\Delta}\right)^2 + \frac{1}{4}\right]} \tag{29}$$

$$= \frac{1}{2} \tag{30}$$

Thus in this case, the ρ dependence cancels out when calculating the probabilities, although we can't expect this to be true in general.