

## RADIALLY SYMMETRIC POTENTIALS, ANGULAR MOMENTUM AND CENTRIFUGAL FORCE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press.  
Chapter 12, Exercise 12.3.5.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

We've seen that the eigenfunctions of two-dimensional angular momentum have the form

$$\psi(\rho, \phi) = R(\rho) \Phi_m(\phi) \quad (1)$$

where

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad (2)$$

In 2 dimensions and polar coordinates, the hamiltonian can be written as

$$H = -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) + V(\rho, \phi) \quad (3)$$

If the potential is radially symmetric, that is, it doesn't depend on  $\phi$ , then

$$H = -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) + V(\rho) \quad (4)$$

In polar coordinates, the angular momentum operator has the form

$$L_z = -i\hbar \frac{\partial}{\partial \phi} \quad (5)$$

Thus  $L_z$  commutes with every term in the hamiltonian 4, so for  $V = V(\rho)$ , we find

$$[H, L_z] = 0 \quad (6)$$

meaning that we can find a set of functions that are simultaneously eigenfunctions of both  $H$  and  $L_z$ . Since we already know what the most general eigenfunctions of  $L_z$  are (eqn 1), the problem is then to find the radial function  $R(\rho)$  so that

$$H[R(\rho)\Phi_m(\phi)] = ER(\rho)\Phi_m(\phi) \quad (7)$$

If we use 4 for  $H$  and 2 for  $\Phi$  we find that we must solve the differential equation

$$-\frac{\hbar^2}{2\mu} \left( \frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} - \frac{m^2}{\rho^2} R \right) + V(\rho)R = ER \quad (8)$$

We've replaced the partial derivatives in 4 by ordinary derivatives, since we now have an ODE in one independent variable, namely  $\rho$ .

The term arising from the  $\frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2}$  term in 4 is similar to a potential term, since it doesn't involve any derivatives of  $R$ . The potential term is

$$V_c = \frac{\hbar^2 m^2}{2\mu \rho^2} \quad (9)$$

We can find the force corresponding to  $V_c$  by taking the negative gradient, which in this case amounts to

$$F_c = -\frac{\partial V_c}{\partial \rho} = \frac{\hbar^2 m^2}{\mu \rho^3} \quad (10)$$

Since the quantum angular momentum is  $\ell_z = m\hbar$ , this can be written as

$$F_c = \frac{\ell_z^2}{\mu \rho^3} \quad (11)$$

If the particle is in a circular orbit, then  $\ell_z = \rho p$  where  $p$  is its momentum, so this becomes

$$F_c = \frac{p^2}{\mu \rho} \quad (12)$$

Classically,  $p = \mu v$  so this is equivalent to

$$F_c = \frac{\mu v^2}{\rho} \quad (13)$$

which is the formula for centrifugal force in Newtonian physics.

PINGBACKS

Pingback: Two-dimensional harmonic oscillator - Part 1