RADIALLY SYMMETRIC POTENTIALS, ANGULAR MOMENTUM AND CENTRIFUGAL FORCE

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.3.5.

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We've seen that the eigenfunctions of two-dimensional angular momentum have the form

$$\psi(\rho,\phi)=R(\rho)\Phi_m(\phi) \tag{1}$$

where

$$\Phi_m(\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi} \tag{2}$$

In 2 dimensions and polar coordinates, the hamiltonian can be written as

$$H=-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)+V(\rho,\phi) \tag{3}$$

If the potential is radially symmetric, that is, it doesn't depend on ϕ, then

$$H=-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)+V(\rho) \tag{4}$$

In polar coordinates, the angular momentum operator has the form

$$L_z=-i\hbar\frac{\partial}{\partial\phi} \tag{5}$$

Thus L_z commutes with every term in the hamiltonian 4, so for $V=V(\rho)$, we find

$$[H,L_z]=0 \tag{6}$$

meaning that we can find a set of functions that are simultaneously eigenfunctions of both H and L_z. Since we already know what the most general eigenfunctions of L_z are (eqn 1), the problem is then to find the radial function $R(\rho)$ so that

$$H\left[R\left(\rho\right)\Phi_{m}\left(\phi\right)\right]=ER\left(\rho\right)\Phi_{m}\left(\phi\right) \tag{7}$$

If we use 4 for H and 2 for Φ we find that we must solve the differential equation

$$-\frac{\hbar^{2}}{2\mu}\left(\frac{d^{2}R}{d\rho^{2}}+\frac{1}{\rho}\frac{dR}{d\rho}-\frac{m^{2}}{\rho^{2}}R\right)+V\left(\rho\right)R=ER \tag{8}$$

We've replaced the partial derivatives in 4 by ordinary derivatives, since we now have an ODE in one independent variable, namely ρ.

The term arising from the $\frac{1}{\rho^{2}}\frac{\partial^{2}}{\partial\phi^{2}}$ term in 4 is similar to a potential term, since it doesn't involve any derivatives of R. The potential term is

$$V_{c}=\frac{\hbar^{2}}{2\mu}\frac{m^{2}}{\rho^{2}} \tag{9}$$

We can find the force corresponding to V_{c} by taking the gradient, which in this case amounts to

$$F_{c}=\frac{\partial V_{c}}{\partial\rho}=-\frac{\hbar^{2}m^{2}}{\mu\rho^{3}} \tag{10}$$

Since the quantum angular momentum is $\ell_{z}=m\hbar$, this can be written as

$$F_{c}=-\frac{\ell_{z}^{2}}{\mu\rho^{3}} \tag{11}$$

If the particle is in a circular orbit, then $\ell_{z}=\rho p$ where p is its momentum, so this becomes

$$F_{c}=-\frac{p^{2}}{\mu\rho} \tag{12}$$

Classically, $p=\mu v^{2}$ so this is equivalent to

$$F_{c}=-\frac{\mu v^{2}}{\rho} \tag{13}$$

which is the formula for centripetal force in Newtonian physics. (Shankar calls it the centrifugal force, but the minus sign indicates it acts towards the centre of rotation rather than outwards, and of course as we well know, the centrifugal force is a fictitious force anyway.)

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