RADIALLY SYMMETRIC POTENTIALS, ANGULAR MOMENTUM AND CENTRIFUGAL FORCE

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.3.5.

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We've seen that the eigenfunctions of two-dimensional angular momentum have the form

$$\psi(\rho,\phi)=R(\rho)\Phi_m(\phi) \tag{1}$$

where

$$\Phi_m(\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi} \tag{2}$$

In 2 dimensions and polar coordinates, the hamiltonian can be written as

$$H=-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)+V(\rho,\phi) \tag{3}$$

If the potential is radially symmetric, that is, it doesn't depend on ϕ, then

$$H=-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)+V(\rho) \tag{4}$$

In polar coordinates, the angular momentum operator has the form

$$L_z=-i\hbar\frac{\partial}{\partial\phi} \tag{5}$$

Thus L_z commutes with every term in the hamiltonian 4, so for $V=V(\rho)$, we find

$$[H,L_z]=0 \tag{6}$$

meaning that we can find a set of functions that are simultaneously eigenfunctions of both H and L_z. Since we already know what the most general eigenfunctions of L_z are (eqn 1), the problem is then to find the radial function $R(\rho)$ so that

$$H\left[R\left(\rho\right)\Phi_{m}\left(\phi\right)\right]=ER\left(\rho\right)\Phi_{m}\left(\phi\right) \tag{7}$$

If we use 4 for H and 2 for Φ we find that we must solve the differential equation

$$-\frac{\hbar^{2}}{2\mu}\left(\frac{d^{2}R}{d\rho^{2}}+\frac{1}{\rho}\frac{dR}{d\rho}-\frac{m^{2}}{\rho^{2}}R\right)+V\left(\rho\right)R=ER \tag{8}$$

We've replaced the partial derivatives in 4 by ordinary derivatives, since we now have an ODE in one independent variable, namely ρ.

The term arising from the $\frac{1}{\rho^{2}}\frac{\partial^{2}}{\partial\phi^{2}}$ term in 4 is similar to a potential term, since it doesn't involve any derivatives of R. The potential term is

$$V_{c}=\frac{\hbar^{2}}{2\mu}\frac{m^{2}}{\rho^{2}} \tag{9}$$

We can find the force corresponding to V_c by taking the negative gradient, which in this case amounts to

$$F_{c}=-\frac{\partial V_{c}}{\partial\rho}=\frac{\hbar^{2}m^{2}}{\mu\rho^{3}} \tag{10}$$

Since the quantum angular momentum is $\ell_{z}=m\hbar$, this can be written as

$$F_{c}=\frac{\ell_{z}^{2}}{\mu\rho^{3}} \tag{11}$$

If the particle is in a circular orbit, then $\ell_{z}=\rho p$ where p is its momentum, so this becomes

$$F_{c}=\frac{p^{2}}{\mu\rho} \tag{12}$$

Classically, $p=\mu v^{2}$ so this is equivalent to

$$F_{c}=\frac{\mu v^{2}}{\rho} \tag{13}$$

which is the formula for centrifugal force in Newtonian physics.

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