

## TWO-DIMENSIONAL HARMONIC OSCILLATOR - PART 1

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.3.7 (1) - (5).

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In this problem, we'll look at solving the 2-dimensional isotropic harmonic oscillator. The solution is fairly lengthy, so we'll split it into two posts, with this being the first. The method of solution is similar to that used in the one-dimensional harmonic oscillator, so you may wish to refer back to that before proceeding.

The Hamiltonian is, in rectangular coordinates:

$$(1) \quad H = \frac{P_x^2 + P_y^2}{2\mu} + \frac{1}{2}\mu\omega^2 (X^2 + Y^2)$$

The potential term is radially symmetric (it doesn't depend on the polar angle  $\phi$ ) so we have a problem of the form considered earlier. We saw there that for such potentials  $[H, L_z] = 0$ . [If you don't believe this, you can grind through the calculations using the commutation relations for  $L_z$  with the rectangular momenta and coordinates, but I won't go through that here.]

As a result,  $L_z$  and  $H$  have simultaneous eigenfunctions of form

$$(2) \quad \psi(\rho, \phi) = R(\rho)\Phi_m(\phi)$$

where

$$(3) \quad \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$

The radial function satisfies the ODE

$$(4) \quad -\frac{\hbar^2}{2\mu} \left( \frac{d^2R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} - \frac{m^2}{\rho^2} R \right) + V(\rho)R = ER$$

where in this case

$$(5) \quad V(\rho) = \frac{1}{2}\mu\omega^2\rho^2$$

Thus the equation we must solve is

$$(6) \quad -\frac{\hbar^2}{2\mu} \left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2} \right) R + \frac{1}{2}\mu\omega^2\rho^2 R = ER$$

To get a feel for the solution, we examine the behaviour in two limiting cases:  $\rho \rightarrow 0$  and  $\rho \rightarrow \infty$ . It's actually easier if we introduce dimensionless variables now, rather than in Shankar's step 4, so we define

$$(7) \quad y \equiv \sqrt{\frac{\mu\omega}{\hbar}}\rho$$

$$(8) \quad \varepsilon \equiv \frac{E}{\hbar\omega}$$

This transforms 6 to

$$(9) \quad -\frac{\hbar^2}{2\mu} \left( \frac{\mu\omega}{\hbar} \right) \left( \frac{d^2}{dy^2} + \frac{1}{y} \frac{d}{dy} - \frac{m^2}{y^2} \right) R + \frac{1}{2}\hbar\omega y^2 R = ER$$

$$(10) \quad -\left( \frac{d^2}{dy^2} + \frac{1}{y} \frac{d}{dy} - \frac{m^2}{y^2} + 2\varepsilon \right) R + y^2 R = 0$$

We can now look at  $y \rightarrow 0$ , and we neglect the terms  $2\varepsilon R$  and  $y^2 R$  to get

$$(11) \quad \left( \frac{d^2}{dy^2} + \frac{1}{y} \frac{d}{dy} - \frac{m^2}{y^2} \right) R = 0$$

If we try a solution of form

$$(12) \quad R = y^{|m|}$$

we have

$$(13) \quad |m|(|m|-1)y^{|m|-2} + |m|y^{|m|-2} - m^2y^{|m|-2} = 0$$

Thus 12 is indeed a solution in this limiting case.

For  $y \rightarrow \infty$ , we can ignore the terms  $\frac{1}{y} \frac{d}{dy}$ ,  $\frac{m^2}{y^2} R$  and  $2\varepsilon R$  to get

$$(14) \quad -\frac{d^2}{dy^2} R + y^2 R = 0$$

or

$$(15) \quad R'' = y^2 R$$

We try a solution of form

$$(16) \quad R = y^a e^{-y^2/2}$$

where  $a$  is some constant. We find

$$(17) \quad R' = (ay^{a-1} - y^{a+1}) e^{-y^2/2}$$

$$(18) \quad R'' = (a(a-1)y^{a-2} - (a+1)y^a - ay^a + y^{a+2}) e^{-y^2/2}$$

$$(19) \quad = y^{a+2} \left( \frac{a(a-1)}{y^4} - \frac{2a+1}{y^2} + 1 \right) e^{-y^2/2}$$

As  $y \rightarrow \infty$ , the last line tends to

$$(20) \quad R'' \rightarrow y^{a+2} e^{-y^2/2} = y^2 R$$

so in this limit 16 is a solution. We can therefore propose that  $R$  has the general form

$$(21) \quad R(y) = y^{|m|} e^{-y^2/2} U(y)$$

where  $U$  is a function to be determined by solving the exact ODE 10. We can get an ODE for  $U$  by substituting 21 into 10, although the calculation gets somewhat messy. As Shankar suggests, we can do this in two stages. First, we substitute

$$(22) \quad R = y^{|m|} f(y)$$

where

$$(23) \quad f(y) = e^{-y^2/2} U(y)$$

The required derivatives are (To make the notation simpler, I'll drop the absolute value signs around  $m$ ; you should assume that wherever  $m$  occurs, it should really be  $|m|$ . We can replace the absolute value sign at the end.)

$$(24) \quad R' = my^{m-1} f + y^m f'$$

$$(25) \quad R'' = m(m-1)y^{m-2} f + 2my^{m-1} f' + y^m f''$$

Plugging these into 10 we have

$$(26) \quad - (m(m-1)y^{m-2}f + 2my^{m-1}f' + y^m f'') - (my^{m-2}f + y^{m-1}f') + \dots$$

$$(27) \quad m^2 y^{m-2} f - 2\epsilon y^m f + y^{m+2} f = 0$$

Collecting terms and dividing through by  $-y^m$ , we get

$$(28) \quad f'' + f' \left( \frac{2m+1}{y} \right) + f(2\epsilon - y^2) = 0$$

We now get the derivatives of  $f$ :

$$(29) \quad f' = -ye^{-y^2/2}U + e^{-y^2/2}U'$$

$$(30) \quad = e^{-y^2/2}(U' - yU)$$

$$(31) \quad f'' = [-y(U' - yU) + U'' - U - yU']e^{-y^2/2}$$

$$(32) \quad = (U'' - 2yU' + (y^2 - 1)U)e^{-y^2/2}$$

When we plug these into 28, the exponential factor cancels out, so we get

$$(33) \quad U'' - 2yU' + (y^2 - 1)U + \frac{2m+1}{y}(U' - yU) + U(2\epsilon - y^2) = 0$$

Collecting terms, we get, upon restoring the absolute values:

$$(34) \quad U'' + \left( \frac{2|m|+1}{y} - 2y \right) U' + (2\epsilon - 2|m| - 2)U = 0$$

We can solve this ODE using a power series in  $y$ , but we'll leave that till the next post.

#### PINGBACKS

Pingback: Two-dimensional harmonic oscillator – Part 2: Series solution

Pingback: Harmonic oscillator in a magnetic field