

TWO-DIMENSIONAL HARMONIC OSCILLATOR – PART 2: SERIES SOLUTION

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press.
Chapter 12, Exercise 12.3.7 (6) - (7).

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In this post, we'll continue with the solution of the 2-d isotropic harmonic oscillator. In the last post, we started with the ODE for the radial function in the form

$$(1) \quad -\frac{\hbar^2}{2\mu} \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2} \right) R + \frac{1}{2} \mu \omega^2 \rho^2 R = ER$$

We introduced dimensionless variables

$$(2) \quad y \equiv \sqrt{\frac{\mu\omega}{\hbar}} \rho$$

$$(3) \quad \varepsilon \equiv \frac{E}{\hbar\omega}$$

and found that R could be written as

$$(4) \quad R(y) = y^{|m|} e^{-y^2/2} U(y)$$

with U given by the solution of the ODE

$$(5) \quad U'' + \left(\frac{2|m|+1}{y} - 2y \right) U' + (2\varepsilon - 2|m| - 2) U = 0$$

We can solve this by using a power series of the form

$$(6) \quad U(y) = \sum_{r=0}^{\infty} C_r y^r$$

where the coefficients C_r are constants.

The derivatives are

$$(7) \quad U' = \sum_{r=0}^{\infty} C_r r y^{r-1}$$

$$(8) \quad = 0 + C_1 + 2C_2 y + 3C_3 y^2 + \dots$$

$$(9) \quad = \sum_{r=0}^{\infty} C_{r+1} (r+1) y^r$$

$$(10) \quad U'' = \sum_{r=0}^{\infty} C_{r+1} r (r+1) y^{r-1}$$

$$(11) \quad = 0 + (1)(2)C_2 + (2)(3)C_3 y + \dots$$

$$(12) \quad = \sum_{r=0}^{\infty} C_{r+2} (r+1)(r+2) y^r$$

Plugging these into 5 we have (we'll drop the absolute value signs on $|m|$ to make the notation simpler; we can restore them at the end):

$$(13) \quad \sum_{r=0}^{\infty} C_{r+2} (r+1)(r+2) y^r + (2m+1) \sum_{r=0}^{\infty} C_r r y^{r-2} - \dots$$

$$(14) \quad 2 \sum_{r=0}^{\infty} C_r r y^r + 2(\varepsilon - m - 1) \sum_{r=0}^{\infty} C_r y^r = 0$$

The second sum in the first line is

$$(15) \quad \sum_{r=0}^{\infty} C_r r y^{r-2} = 0 + C_1 y^{-1} + 2C_2 + 3C_3 y + \dots$$

$$(16) \quad = \sum_{r=-1}^{\infty} C_{r+2} (r+2) y^r$$

The sum thus becomes

$$(17) \quad (2m+1)C_1 y^{-1} + \sum_{r=0}^{\infty} y^r C_{r+2} (r+2)^2 + 2 \sum_{r=0}^{\infty} y^r C_r [-r + \varepsilon - m - 1] = 0$$

A basic theorem about power series is that if the sum of the series equals zero for all y , then the coefficient of each power must be zero. This shows that $C_1 = 0$ since otherwise the series would blow up as $y \rightarrow 0$. This results in a recursion relation for the C_r :

$$(18) \quad C_{r+2} = \frac{2(r+m+1-\varepsilon)}{(r+2)^2} C_r$$

Since $C_1 = 0$, all $C_r = 0$ for odd r . For large r we have

$$(19) \quad \frac{C_{r+2}}{C_r} \rightarrow \frac{2}{r}$$

If the series is allowed to be infinite, this leads to a divergent series as we can see from the following (based on Shankar's section 7.3). Suppose we look at $y^m e^{y^2}$, which clearly goes to infinity at large y (remember, m is positive). In series form this is

$$(20) \quad y^m e^{y^2} = \sum_{k=0}^{\infty} \frac{y^{2k+m}}{k!}$$

The coefficient C_n of y^n , with $n = 2k + m$ in this series is

$$(21) \quad C_n = \frac{1}{[(n-m)/2]!}$$

Similarly,

$$(22) \quad C_{n+2} = \frac{1}{[(n+2-m)/2]!}$$

The ratio is

$$(23) \quad \frac{C_{n+2}}{C_n} = \frac{[(n-m)/2]!}{[(n+2-m)/2]!}$$

$$(24) \quad = \frac{1}{(n-m)/2 + 1}$$

$$(25) \quad \rightarrow \frac{2}{n}$$

In other words, the coefficients of our series solution have the same behaviour 19 for large r as those in the series for $y^m e^{y^2}$. Referring back to 4, we see that this gives an overall behaviour for the radial function R of

$$(26) \quad R \rightarrow y^m e^{-y^2/2} y^m e^{y^2} = y^{2m} e^{y^2/2}$$

Thus if we allow the series for U to be infinite, the overall solution diverges, which is not acceptable. We therefore require that the series terminates at some finite value of r , and from 18 we see that this happens if

$$(27) \quad \varepsilon = r + m + 1$$

for some r . From the definition 3 this gives us the allowed values for the energy:

$$(28) \quad E = \hbar\omega(r + |m| + 1)$$

$$(29) \quad = \hbar\omega(2k + |m| + 1)$$

where the last line follows because r must be even. If

$$(30) \quad n \equiv 2k + |m|$$

then the allowed energies are

$$(31) \quad E = \hbar\omega(n + 1)$$

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