TWO-DIMENSIONAL HARMONIC OSCILLATOR – PART 2: SERIES SOLUTION

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.3.7 (6) - (7).

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In this post, we'll continue with the solution of the 2-d isotropic harmonic oscillator. In the last post, we started with the ODE for the radial function in the form

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m^2}{\rho^2}\right)R+\frac{1}{2}\mu\omega^2\rho^2R=ER \tag{1}$$

We introduced dimensionless variables

$$y \equiv \sqrt{\frac{\mu\omega}{\hbar}}\rho \tag{2}$$

$$\varepsilon \equiv \frac{E}{\hbar\omega} \tag{3}$$

and found that R could be written as

$$R(y)=y^{|m|}e^{-y^2/2}U(y) \tag{4}$$

with U given by the solution of the ODE

$$U''+\left(\frac{2|m|+1}{y}-2y\right)U'+(2\varepsilon-2|m|-2)U=0 \tag{5}$$

We can solve this by using a power series of the form

$$U(y)=\sum_{r=0}^{\infty}C_ry^r \tag{6}$$

where the coefficients C_r are constants.

The derivatives are

$$U' = \sum_{r=0}^{\infty} C_r r y^{r-1} \tag{7}$$

$$= 0 + C_1 + 2C_2 y + 3C_3 + y^2 + \ldots \tag{8}$$

$$= \sum_{r=0}^{\infty} C_{r+1} (r+1) y^r \tag{9}$$

$$U'' = \sum_{r=0}^{\infty} C_{r+1} r (r+1) y^{r-1} \tag{10}$$

$$= 0 + (1)(2) C_2 + (2)(3) C_3 y + \ldots \tag{11}$$

$$= \sum_{r=0}^{\infty} C_{r+2} (r+1)(r+2) y^r \tag{12}$$

Plugging these into 5 we have (we'll drop the absolute value signs on $|m|$ to make the notation simpler; we can restore them at the end):

$$\sum_{r=0}^{\infty} C_{r+2} (r+1)(r+2) y^r + (2m+1) \sum_{r=0}^{\infty} C_r r y^{r-2} - \ldots \tag{13}$$

$$2 \sum_{r=0}^{\infty} C_r r y^r + 2(\varepsilon - m - 1) \sum_{r=0}^{\infty} C_r y^r = 0 \tag{14}$$

The second sum in the first line is

$$\sum_{r=0}^{\infty} C_r r y^{r-2} = 0 + C_1 y^{-1} + 2C_2 + 3C_3 y + \ldots \tag{15}$$

$$= \sum_{r=-1}^{\infty} C_{r+2} (r+2) y^r \tag{16}$$

The sum thus becomes

$$(2m+1) C_1 y^{-1} + \sum_{r=0}^{\infty} y^r C_{r+2} (r+2)^2 + 2 \sum_{r=0}^{\infty} y^r C_r [-r + \varepsilon - m - 1] = 0 \tag{17}$$

A basic theorem about power series is that if the sum of the series equals zero for all y, then the coefficient of each power must be zero. This shows that $C_1 = 0$ since otherwise the series would blow up as $y \to 0$. This results in a recursion relation for the C_r:

$$C_{r+2} = \frac{2(r + m + 1 - \varepsilon)}{(r+2)^2} C_r \tag{18}$$

Since $C_1 = 0$, all $C_r = 0$ for odd r. For large r we have

$$\frac{C_{r+2}}{C_r} \rightarrow \frac{2}{r} \tag{19}$$

If the series is allowed to be infinite, this leads to a divergent series as we can see from the following (based on Shankar's section 7.3). Suppose we look at $y^m e^{y^2}$, which clearly goes to infinity at large y (remember, m is positive). In series form this is

$$y^m e^{y^2} = \sum_{k=0}^{\infty} \frac{y^{2k+m}}{k!} \tag{20}$$

The coefficient C_n of y^n , with $n = 2k+m$ in this series is

$$C_n = \frac{1}{[(n-m)/2]!} \tag{21}$$

Similarly,

$$C_{n+2} = \frac{1}{[(n+2-m)/2]!} \tag{22}$$

The ratio is

$$\begin{aligned} \frac{C_{n+2}}{C_n} &= \frac{[(n-m)/2]!}{[(n+2-m)/2]!} && (23) \\ &= \frac{1}{(n-m)/2+1} && (24) \\ &\rightarrow \frac{2}{n} && (25) \end{aligned}$$

In other words, the coefficients of our series solution have the same behaviour 19 for large r as those in the series for $y^m e^{y^2}$. Referring back to 4, we see that this gives an overall behaviour for the radial function R of

$$R \rightarrow y^m e^{-y^2/2} y^m e^{y^2} = y^{2m} e^{y^2/2} \tag{26}$$

Thus if we allow the series for U to be infinite, the overall solution diverges, which is not acceptable. We therefore require that the series terminates at some finite value of r, and from 18 we see that this happens if

$$\varepsilon = r + m + 1 \tag{27}$$

for some r. From the definition 3 this gives us the allowed values for the energy:

$$E = \hbar\omega\left(r+|m|+1\right) \tag{28}$$
$$= \hbar\omega\left(2k+|m|+1\right) \tag{29}$$

where the last line follows because r must be even. If

$$n \equiv 2k+|m| \tag{30}$$

then the allowed energies are

$$E = \hbar\omega\left(n+1\right) \tag{31}$$

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