

HARMONIC OSCILLATOR IN 2 DIMENSIONS: COMPARISON WITH RECTANGULAR COORDINATES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.3.7 (8) - (10).

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In this post, we'll continue with the solution of the 2-d isotropic harmonic oscillator. In the previous post, we found that the radial equation R can be written as

$$(0.1) \quad R(y) = y^{|m|} e^{-y^2/2} U(y)$$

where the dimensionless variables are given by

$$(0.2) \quad y \equiv \sqrt{\frac{\mu\omega}{\hbar}} \rho$$

$$(0.3) \quad \varepsilon \equiv \frac{E}{\hbar\omega}$$

and U has a solution as a power series

$$(0.4) \quad U(y) = \sum_{r=0}^{\infty} C_r y^r$$

The coefficients C_r satisfy the recursion relation

$$(0.5) \quad C_{r+2} = \frac{2(r + |m| + 1 - \varepsilon)}{(r + 2)^2} C_r$$

Only C_r for even r are non-zero.

In order for U to remain finite for large y , the series must terminate, which gives the allowable values for the energy as

$$(0.6) \quad E = \hbar\omega(n + 1)$$

with

$$(0.7) \quad n \equiv 2k + |m|$$

and $k = 0, 1, 2, \dots$

We can now compare the solution obtained in polar coordinates with our earlier solution in terms of rectangular coordinates. First, what are the possible values for m for a given energy $E = \hbar\omega(n+1)$? From the relation 0.7, we can look at even and odd n separately. For even n , k can take values $0, 1, \dots, \frac{n}{2} - 1, \frac{n}{2}$. The first $\frac{n}{2}$ of these values for k (that is, for $k = 0, 1, \dots, \frac{n}{2} - 1$) each allow two values of m such that $|m| = n - 2k$, namely $m = \pm(n - 2k)$. If $k = \frac{n}{2}$, then we must have $m = 0$. Thus for even n the total number of combinations is $2 \times \frac{n}{2} + 1 = n + 1$.

For odd n , k can take on values $0, 1, \dots, \frac{n-1}{2}$, giving a total of $\frac{n+1}{2}$ possible values. (If this isn't obvious, write it out for the first few values of odd n to see the pattern.) For each of these values of k , m can take on the two values $m = \pm(n - 2k)$, thus there are again $2 \times \frac{n+1}{2} = n + 1$ different combinations. Thus a state with energy $E = \hbar\omega(n+1)$ has a degeneracy $n + 1$.

We can construct the actual eigenfunctions for a couple of values of n by plugging in the appropriate formulas. For $n = 0$ there is only one function, which we find by setting $k = m = 0$. From 0.4, we have

$$(0.8) \quad U_0(y) = C_0$$

and from 0.1 we have

$$(0.9) \quad R_0(y) = C_0 e^{-y^2/2}$$

or, in terms of the original variables

$$(0.10) \quad R_0(\rho) = C_0 e^{-\mu\omega\rho^2/2\hbar}$$

The complete solution is given by

$$(0.11) \quad \psi_m(\rho, \phi) = R(\rho) \Phi_m(\phi)$$

$$(0.12) \quad = \frac{1}{\sqrt{2\pi}} R(\rho) e^{im\phi}$$

so for $m = 0$ we have

$$(0.13) \quad \psi_0(\rho, \phi) = \frac{C_0}{\sqrt{2\pi}} e^{-\mu\omega\rho^2/2\hbar}$$

The constant C_0 can be found by normalizing:

$$(0.14) \quad 1 = \int_0^\infty \int_0^{2\pi} |\psi_0|^2 \rho d\phi d\rho$$

$$(0.15) \quad = |C_0|^2 \int_0^\infty e^{-\mu\omega\rho^2/\hbar} \rho d\rho$$

$$(0.16) \quad = |C_0|^2 \frac{\hbar}{2\mu\omega}$$

$$(0.17) \quad C_0 = \sqrt{\frac{2\mu\omega}{\hbar}}$$

$$(0.18) \quad \psi_0(\rho, \phi) = \sqrt{\frac{\mu\omega}{\pi\hbar}} e^{-\mu\omega\rho^2/2\hbar}$$

This agrees with the earlier result in rectangular coordinates (eqn 26 in this post). This must be the case, since the $n = 0$ state is non-degenerate.

For $n = 1$, we have $k = 0$ and $m = \pm 1$ so we have two solutions:

$$(0.19) \quad \psi_1 = \frac{C_0}{\sqrt{2\pi}} \sqrt{\frac{\mu\omega}{\hbar}} \rho e^{-\mu\omega\rho^2/2\hbar} e^{i\phi}$$

$$(0.20) \quad \psi_{-1} = \frac{C_0}{\sqrt{2\pi}} \sqrt{\frac{\mu\omega}{\hbar}} \rho e^{-\mu\omega\rho^2/2\hbar} e^{-i\phi}$$

Again, we normalize

$$(0.21) \quad 1 = \int_0^\infty \int_0^{2\pi} |\psi_{\pm 1}|^2 \rho d\phi d\rho$$

$$(0.22) \quad = \frac{\mu\omega}{\hbar} |C_0|^2 \int_0^\infty e^{-\mu\omega\rho^2/\hbar} \rho^3 d\rho$$

$$(0.23) \quad C_0 = \sqrt{\frac{2\mu\omega}{\hbar}}$$

$$(0.24) \quad \psi_{\pm 1} = \frac{\mu\omega}{\hbar\sqrt{\pi}} \rho e^{-\mu\omega\rho^2/2\hbar} e^{\pm i\phi}$$

These solutions are linear combinations of the corresponding solutions in rectangular coordinates:

$$(0.25) \quad \psi_{10} = \frac{\sqrt{2}\mu\omega}{\hbar\sqrt{\pi}} e^{-\mu\omega\rho^2/2\hbar} \rho \cos \phi$$

$$(0.26) \quad \psi_{01} = \frac{\sqrt{2}\mu\omega}{\hbar\sqrt{\pi}} e^{-\mu\omega\rho^2/2\hbar} \rho \sin \phi$$

The combinations are

$$(0.27) \quad \psi_{+1} = \frac{1}{\sqrt{2}}(\psi_{10} + i\psi_{01})$$

$$(0.28) \quad \psi_{-1} = \frac{1}{\sqrt{2}}(\psi_{10} - i\psi_{01})$$

The parity of the states is found from their behaviour under the transformation (in rectangular coordinates) $x \rightarrow -x$ and $y \rightarrow -y$. In polar coordinates this is equivalent to the transformation $\phi \rightarrow \phi + \pi$ and from 0.18 and 0.24 we see that

$$(0.29) \quad \psi_0(\rho, \phi + \pi) = \psi_0(\rho, \phi)$$

$$(0.30) \quad \psi_{\pm 1}(\rho, \phi + \pi) = \psi_{\pm 1}(\rho, \phi) e^{\pm \pi}$$

$$(0.31) \quad = -\psi_{\pm 1}(\rho, \phi)$$

Thus the parity of $n = 0$ is even, and that of $n = 1$ is odd. In general, since the ϕ dependence enters only through the term $e^{im\phi} = e^{in\phi} e^{-2ik\phi}$, we see that adding π to ϕ leaves the $e^{-2ik\phi}$ term unchanged and multiplies the $e^{in\phi}$ term by $e^{in\pi} = (-1)^n$, so the parity of state n is $(-1)^n$.

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