HARMONIC OSCILLATOR IN A MAGNETIC FIELD

As another example of the harmonic oscillator, we’ll look at a charged particle moving in a magnetic field. The field \( B \) is given in terms of the magnetic vector potential

\[
A = \frac{B}{2} (-y \hat{x} + x \hat{y})
\]  

(1)

The field is

\[
B = \nabla \times A
\]  

(2)

\[
= \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z}
\]  

(3)

\[
= B \hat{z}
\]  

(4)

If the particle is confined to the \( xy \) plane and the magnetic field provides the only force, the force is given by the Lorentz force law

\[
F = qv \times B
\]  

(5)

Since \( F \) is always perpendicular to the direction of motion \( v \), the magnetic force does no work, so the kinetic energy and hence the speed \( v \) of the particle is constant. Classically, the particle is thus confined to move in a circle with \( F \) providing the centripetal force, so we have

\[
qvB = \frac{\mu v^2}{\rho}
\]  

(6)

\[
v = \frac{qB\rho}{\mu}
\]  

(7)

where \( q \) is the charge, \( \mu \) is the mass and \( \rho \) is the radius of the circle. The period of the orbit is
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\[ T = \frac{2\pi \rho}{v} = \frac{2\pi \mu}{qB} \]  

which gives an angular frequency of

\[ \omega_0 = \frac{2\pi}{T} = \frac{qB}{\mu} \]  

This is the result in SI units; Shankar uses Gaussian units, in which the magnetic field picks up a factor of \( \frac{1}{c} \), so in Shankar’s notation, this is

\[ \omega_0 = \frac{qB}{\mu c} \]  

As the rest of the problem relies on Gaussian units, we’ll stick to them from now on.

Classically, the Hamiltonian for the electromagnetic force is

\[ H = \frac{|p - qa/c|^2}{2\mu} + q\phi \]  

where \( \phi \) is the electric potential, which is zero here. Thus using (1) we have for the quantum version in which \( p \) and the position vector are replaced by operators

\[ H = \left( \frac{P_x + qYB/2c}{2\mu} \right)^2 + \left( \frac{P_y - qXB/2c}{2\mu} \right)^2 \]  

We can perform a canonical transformation by defining

\[ Q \equiv \frac{1}{qB} \left( cP_x + \frac{qYB}{2} \right) \]  
\[ P \equiv P_y - \frac{qXB}{2c} \]  

We can verify that these coordinates are canonical by checking their commutator:

\[ [Q, P] = \frac{1}{qB} \left[ cP_x + \frac{qYB}{2}, P_y - \frac{qXB}{2c} \right] \]  
\[ = \frac{1}{qB} \left( -\frac{qB}{2} [P_x, X] + \frac{qB}{2} [Y, P_y] \right) \]  
\[ = \frac{i\hbar}{2} \]  
\[ = i\hbar \]
Thus $Q$ and $P$ have the correct commutator for a pair of position and momentum variables. 

Rewriting 12 in terms of $Q$ and $P$, we have

$$H = \frac{q^2 B^2}{2\mu c^2} Q^2 + \frac{p^2}{2\mu}$$  \hspace{1cm} (19)$$

$$= \frac{p^2}{2\mu} + \frac{\mu}{2} \omega_0^2 Q^2$$  \hspace{1cm} (20)$$

Thus $H$ has the same form as that for a one-dimensional harmonic oscillator with frequency $\omega_0$, so the energy levels of this system must be

$$E = \left(n + \frac{1}{2}\right) \hbar \omega_0$$  \hspace{1cm} (21)$$

We can expand 12 in terms of the original position and momentum variables to get

$$H = \frac{p_x^2 + p_y^2}{2\mu} + \frac{1}{2\mu} \left(\frac{qB}{2\mu c}\right)^2 (X^2 + Y^2) + \frac{qB}{2\mu c} (P_x Y - P_y X)$$  \hspace{1cm} (22)$$

$$= \frac{p_x^2 + p_y^2}{2\mu} + \frac{1}{2\mu} \left(\frac{\omega_0}{2}\right)^2 (X^2 + Y^2) - \frac{\omega_0}{2} (X P_y - Y P_x)$$  \hspace{1cm} (23)$$

$$= H \left(\frac{\omega_0}{2}, \mu\right) - \frac{\omega_0}{2} L_z$$  \hspace{1cm} (24)$$

where $H \left(\frac{\omega_0}{2}, \mu\right)$ is the Hamiltonian for a 2-dim harmonic oscillator with frequency $\omega_0/2$. As we saw when solving that system, the Hamiltonian for the isotropic oscillator commutes with $L_z$ since the potential is radially symmetric, thus the eigenfunctions of $H$ are also eigenfunctions of $L_z$. In terms of the present problem, this means that the eigenfunctions of $H \left(\frac{\omega_0}{2}, \mu\right)$ are also eigenfunctions of $L_z$ and thus also eigenfunctions of $H$. In our solution of the 2-dim isotropic oscillator, we found that the energy levels are given by

$$E = \hbar \omega \left(2k + |m| + 1\right)$$  \hspace{1cm} (25)$$

where $k = 0, 1, 2, \ldots$ and $m$ is the angular momentum (in units of $\hbar$). Thus for the oscillator with Hamiltonian $H \left(\frac{\omega_0}{2}, \mu\right)$, the energy levels are

$$E = \frac{1}{2} \hbar \omega_0 \left(2k + |m| + 1\right)$$  \hspace{1cm} (26)$$

$$= \hbar \omega_0 \left(k + \frac{1}{2} |m| + \frac{1}{2}\right)$$  \hspace{1cm} (27)$$
The energy levels of the original $H$ are therefore, from (24)

$$E = \hbar \omega_0 \left( k + \frac{1}{2} |m| + \frac{1}{2} \right) - \frac{\omega_0}{2} m \hbar$$

(28)

$$= \hbar \omega_0 \left( k + \frac{1}{2} |m| - \frac{1}{2} m + \frac{1}{2} \right)$$

(29)

[Shankar says the $k$ can be 'any integer', but from our original derivation of (25) we found that $k$ is a non-negative integer.] Equation (29) gives the same energies as (21), since if $m > 0$, we get $E = \hbar \omega_0 \left( k + \frac{1}{2} \right)$, while if $m < 0$ we have $E = \hbar \omega_0 \left( k + |m| + \frac{1}{2} \right)$. Both $k + \frac{1}{2}$ and $k + |m| + \frac{1}{2}$ give the same sequence of values as $n + \frac{1}{2}$. [I’m not quite sure the two methods are equivalent, though, since (21) being the solution of a one-dimensional system is non-degenerate, while (29) being a two-dimensional system does have degenerate energy levels.]