HARMONIC OSCILLATOR IN A MAGNETIC FIELD

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.3.8.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

As another example of the harmonic oscillator, we'll look at a charged particle moving in a magnetic field. The field $\mathbf{B}$ is given in terms of the magnetic vector potential

$$\mathbf{A}=\frac{B}{2}\left(-y\hat{\mathbf{x}}+x\hat{\mathbf{y}}\right) \tag{1}$$

The field is

$$\begin{aligned}
\mathbf{B} &= \nabla\times\mathbf{A} && (2)\\
&= \left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}} && (3)\\
&= B\hat{\mathbf{z}} && (4)
\end{aligned}$$

If the particle is confined to the xy plane and the magnetic field provides the only force, the force is given by the Lorentz force law

$$\mathbf{F}=q\mathbf{v}\times\mathbf{B} \tag{5}$$

Since $\mathbf{F}$ is always perpendicular to the direction of motion $\mathbf{v}$, the magnetic force does no work, so the kinetic energy and hence the speed v of the particle is constant. Classically, the particle is thus confined to move in a circle with $\mathbf{F}$ providing the centripetal force, so we have

$$\begin{aligned}
qvB &= \frac{\mu v^2}{\rho} && (6)\\
v &= \frac{qB\rho}{\mu} && (7)
\end{aligned}$$

where q is the charge, μ is the mass and ρ is the radius of the circle. The period of the orbit is

$$T=\frac{2\pi\rho}{v}=\frac{2\pi\mu}{qB} \tag{8}$$

which gives an angular frequency of

$$\omega_0=\frac{2\pi}{T}=\frac{qB}{\mu} \tag{9}$$

This is the result in SI units; Shankar uses Gaussian units, in which the magnetic field picks up a factor of $\frac{1}{c}$, so in Shankar's notation, this is

$$\omega_0=\frac{qB}{\mu c} \tag{10}$$

As the rest of the problem relies on Gaussian units, we'll stick to them from now on.

Classically, the Hamiltonian for the electromagnetic force is

$$H=\frac{|\mathbf{p}-q\mathbf{A}/c|^2}{2\mu}+q\phi \tag{11}$$

where ϕ is the electric potential, which is zero here. Thus using 1, we have for the quantum version in which $\mathbf{p}$ and the position vector are replaced by operators

$$H=\frac{(P_x+qYB/2c)^2}{2\mu}+\frac{(P_y-qXB/2c)^2}{2\mu} \tag{12}$$

We can perform a canonical transformation by defining

$$Q \equiv \frac{1}{qB}\left(cP_x+\frac{qYB}{2}\right) \tag{13}$$

$$P \equiv P_y-\frac{qXB}{2c} \tag{14}$$

We can verify that these coordinates are canonical by checking their commutator:

$$[Q,P] = \frac{1}{qB}\left[cP_x + \frac{qYB}{2}, P_y - \frac{qXB}{2c}\right] \tag{15}$$

$$= \frac{1}{qB}\left(-\frac{qB}{2}[P_x,X] + \frac{qB}{2}[Y,P_y]\right) \tag{16}$$

$$= \frac{i\hbar}{2} + \frac{i\hbar}{2} \tag{17}$$

$$= i\hbar \tag{18}$$

Thus Q and P have the correct commutator for a pair of position and momentum variables.

Rewriting 12 in terms of Q and P, we have

$$H = \frac{q^2B^2}{2\mu c^2}Q^2 + \frac{P^2}{2\mu} \tag{19}$$

$$= \frac{P^2}{2\mu} + \frac{\mu}{2}\omega_0^2 Q^2 \tag{20}$$

Thus H has the same form as that for a one-dimensional harmonic oscillator with frequency ω_0, so the energy levels of this system must be

$$E = \left(n + \frac{1}{2}\right)\hbar\omega_0 \tag{21}$$

We can expand 12 in terms of the original position and momentum variables to get

$$H = \frac{P_x^2 + P_y^2}{2\mu} + \frac{1}{2}\mu\left(\frac{qB}{2\mu c}\right)^2\left(X^2+Y^2\right) + \frac{qB}{2\mu c}\left(P_xY - P_yX\right) \tag{22}$$

$$= \frac{P_x^2 + P_y^2}{2\mu} + \frac{1}{2}\mu\left(\frac{\omega_0}{2}\right)^2\left(X^2+Y^2\right) - \frac{\omega_0}{2}\left(XP_y - YP_x\right) \tag{23}$$

$$= H\left(\frac{\omega_0}{2},\mu\right) - \frac{\omega_0}{2}L_z \tag{24}$$

where $H\left(\frac{\omega_0}{2},\mu\right)$ is the hamiltonian for a 2-dim harmonic oscillator with frequency $\omega_0/2$. As we saw when solving that system, the Hamiltonian for the isotropic oscillator commutes with L_z since the potential is radially symmetric, thus the eigenfunctions of H are also eigenfunctions of L_z. In terms of the present problem, this means that the eigenfunctions of $H\left(\frac{\omega_0}{2},\mu\right)$ are also eigenfunctions of L_z and thus also eigenfunctions of H. In our solution of the 2-dim isotropic oscillator, we found that the energy levels are given by

$$E = \hbar\omega\left(2k + |m| + 1\right) \tag{25}$$

where $k = 0, 1, 2, \ldots$ and m is the angular momentum (in units of $\hbar$). Thus for the oscillator with Hamiltonian $H\left(\frac{\omega_0}{2}, \mu\right)$, the energy levels are

$$\begin{aligned} E &= \frac{1}{2}\hbar\omega_0\left(2k + |m| + 1\right) && (26) \\ &= \hbar\omega_0\left(k + \frac{1}{2}|m| + \frac{1}{2}\right) && (27) \end{aligned}$$

The energy levels of the original H are therefore, from 24

$$\begin{aligned} E &= \hbar\omega_0\left(k + \frac{1}{2}|m| + \frac{1}{2}\right) - \frac{\omega_0}{2}m\hbar && (28) \\ &= \hbar\omega_0\left(k + \frac{1}{2}|m| - \frac{1}{2}m + \frac{1}{2}\right) && (29) \end{aligned}$$

[Shankar says the k can be 'any integer', but from our original derivation of 25, we found that k is a non-negative integer.] Equation 29 gives the same energies as 21, since if $m > 0$, we get $E = \hbar\omega_0\left(k + \frac{1}{2}\right)$, while if $m < 0$ we have $E = \hbar\omega_0\left(k + |m| + \frac{1}{2}\right)$. Both $k + \frac{1}{2}$ and $k + |m| + \frac{1}{2}$ give the same sequence of values as $n + \frac{1}{2}$. [I'm not quite sure the two methods are equivalent, though, since 21, being the solution of a one-dimensional system is non-degenerate, while 29, being a two-dimensional system does have degenerate energy levels.]