

## HARMONIC OSCILLATOR IN A MAGNETIC FIELD

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.3.8.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

As another example of the harmonic oscillator, we'll look at a charged particle moving in a magnetic field. The field  $\mathbf{B}$  is given in terms of the magnetic vector potential

$$(0.1) \quad \mathbf{A} = \frac{B}{2} (-y\hat{\mathbf{x}} + x\hat{\mathbf{y}})$$

The field is

$$(0.2) \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$(0.3) \quad = \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{\mathbf{z}}$$

$$(0.4) \quad = B\hat{\mathbf{z}}$$

If the particle is confined to the  $xy$  plane and the magnetic field provides the only force, the force is given by the Lorentz force law

$$(0.5) \quad \mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

Since  $\mathbf{F}$  is always perpendicular to the direction of motion  $\mathbf{v}$ , the magnetic force does no work, so the kinetic energy and hence the speed  $v$  of the particle is constant. Classically, the particle is thus confined to move in a circle with  $\mathbf{F}$  providing the centripetal force, so we have

$$(0.6) \quad qvB = \frac{\mu v^2}{\rho}$$

$$(0.7) \quad v = \frac{qB\rho}{\mu}$$

where  $q$  is the charge,  $\mu$  is the mass and  $\rho$  is the radius of the circle. The period of the orbit is

$$(0.8) \quad T = \frac{2\pi\rho}{v} = \frac{2\pi\mu}{qB}$$

which gives an angular frequency of

$$(0.9) \quad \omega_0 = \frac{2\pi}{T} = \frac{qB}{\mu}$$

This is the result in SI units; Shankar uses Gaussian units, in which the magnetic field picks up a factor of  $\frac{1}{c}$ , so in Shankar's notation, this is

$$(0.10) \quad \omega_0 = \frac{qB}{\mu c}$$

As the rest of the problem relies on Gaussian units, we'll stick to them from now on.

Classically, the Hamiltonian for the electromagnetic force is

$$(0.11) \quad H = \frac{|\mathbf{p} - q\mathbf{A}/c|^2}{2\mu} + q\phi$$

where  $\phi$  is the electric potential, which is zero here. Thus using 0.1, we have for the quantum version in which  $\mathbf{p}$  and the position vector are replaced by operators

$$(0.12) \quad H = \frac{(P_x + qYB/2c)^2}{2\mu} + \frac{(P_y - qXB/2c)^2}{2\mu}$$

We can perform a canonical transformation by defining

$$(0.13) \quad Q \equiv \frac{1}{qB} \left( cP_x + \frac{qYB}{2} \right)$$

$$(0.14) \quad P \equiv P_y - \frac{qXB}{2c}$$

We can verify that these coordinates are canonical by checking their commutator:

$$(0.15) \quad [Q, P] = \frac{1}{qB} \left[ cP_x + \frac{qYB}{2}, P_y - \frac{qXB}{2c} \right]$$

$$(0.16) \quad = \frac{1}{qB} \left( -\frac{qB}{2} [P_x, X] + \frac{qB}{2} [Y, P_y] \right)$$

$$(0.17) \quad = \frac{i\hbar}{2} + \frac{i\hbar}{2}$$

$$(0.18) \quad = i\hbar$$

Thus  $Q$  and  $P$  have the correct commutator for a pair of position and momentum variables.

Rewriting 0.12 in terms of  $Q$  and  $P$ , we have

$$(0.19) \quad H = \frac{q^2 B^2}{2\mu c^2} Q^2 + \frac{P^2}{2\mu}$$

$$(0.20) \quad = \frac{P^2}{2\mu} + \frac{\mu}{2} \omega_0^2 Q^2$$

Thus  $H$  has the same form as that for a one-dimensional harmonic oscillator with frequency  $\omega_0$ , so the energy levels of this system must be

$$(0.21) \quad E = \left( n + \frac{1}{2} \right) \hbar \omega_0$$

We can expand 0.12 in terms of the original position and momentum variables to get

$$(0.22) \quad H = \frac{P_x^2 + P_y^2}{2\mu} + \frac{1}{2}\mu \left( \frac{qB}{2\mu c} \right)^2 (X^2 + Y^2) + \frac{qB}{2\mu c} (P_x Y - P_y X)$$

$$(0.23) \quad = \frac{P_x^2 + P_y^2}{2\mu} + \frac{1}{2}\mu \left( \frac{\omega_0}{2} \right)^2 (X^2 + Y^2) - \frac{\omega_0}{2} (X P_y - Y P_x)$$

$$(0.24) \quad = H \left( \frac{\omega_0}{2}, \mu \right) - \frac{\omega_0}{2} L_z$$

where  $H \left( \frac{\omega_0}{2}, \mu \right)$  is the hamiltonian for a 2-dim harmonic oscillator with frequency  $\omega_0/2$ . As we saw when solving that system, the Hamiltonian for the isotropic oscillator commutes with  $L_z$  since the potential is radially symmetric, thus the eigenfunctions of  $H$  are also eigenfunctions of  $L_z$ . In terms of the present problem, this means that the eigenfunctions of  $H \left( \frac{\omega_0}{2}, \mu \right)$  are also eigenfunctions of  $L_z$  and thus also eigenfunctions of  $H$ . In our solution of the 2-dim isotropic oscillator, we found that the energy levels are given by

$$(0.25) \quad E = \hbar\omega (2k + |m| + 1)$$

where  $k = 0, 1, 2, \dots$  and  $m$  is the angular momentum (in units of  $\hbar$ ). Thus for the oscillator with Hamiltonian  $H\left(\frac{\omega_0}{2}, \mu\right)$ , the energy levels are

$$(0.26) \quad E = \frac{1}{2}\hbar\omega_0 (2k + |m| + 1)$$

$$(0.27) \quad = \hbar\omega_0 \left(k + \frac{1}{2}|m| + \frac{1}{2}\right)$$

The energy levels of the original  $H$  are therefore, from 0.24

$$(0.28) \quad E = \hbar\omega_0 \left(k + \frac{1}{2}|m| + \frac{1}{2}\right) - \frac{\omega_0}{2}m\hbar$$

$$(0.29) \quad = \hbar\omega_0 \left(k + \frac{1}{2}|m| - \frac{1}{2}m + \frac{1}{2}\right)$$

[Shankar says the  $k$  can be 'any integer', but from our original derivation of 0.25, we found that  $k$  is a non-negative integer.] Equation 0.29 gives the same energies as 0.21, since if  $m > 0$ , we get  $E = \hbar\omega_0 \left(k + \frac{1}{2}\right)$ , while if  $m < 0$  we have  $E = \hbar\omega_0 \left(k + |m| + \frac{1}{2}\right)$ . Both  $k + \frac{1}{2}$  and  $k + |m| + \frac{1}{2}$  give the same sequence of values as  $n + \frac{1}{2}$ . [I'm not quite sure the two methods are equivalent, though, since 0.21, being the solution of a one-dimensional system is non-degenerate, while 0.29, being a two-dimensional system does have degenerate energy levels.]