LEVI-CIVITA ANTISYMMETRIC TENSOR, VECTOR PRODUCTS AND SYSTEMS OF 3 FERMIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.4.1.

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The Levi-Civita symbol ε_{ijk} is defined as $+1$ if i,j,k have the values 1,2,3 (in that order), 2,3,1 or 3,1,2. Swapping any pair of indices multiplies the value by -1, so that, for example, $\varepsilon_{123}=+1$ and $\varepsilon_{213}=-1$. If two indices are the same, such as $i=j=1$, then swapping them leaves ε_{11k} unchanged so the requirement that $\varepsilon_{ijk}=-\varepsilon_{jik}$ means that $\varepsilon_{ijk}=0$ if any two of its indices are equal.

The symbol is actually an antisymmetric tensor of rank 3, and is found frequently in physical and mathematical equations. One example is in the cross product of two 3-d vectors. If

$$\mathbf{c}=\mathbf{a}\times\mathbf{b} \tag{1}$$

we can work out the components of $\mathbf{c}$ in the usual way by calculating the determinant:

$$\begin{aligned}\mathbf{c} &= \begin{vmatrix}\hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} & (2)\\ &= (a_2b_3-b_2a_3)\,\hat{\mathbf{x}}_1-(a_1b_3-b_1a_3)\,\hat{\mathbf{x}}_2+(a_1b_2-b_2a_1)\,\hat{\mathbf{x}}_3 & (3)\end{aligned}$$

where I've used $\hat{\mathbf{x}}_1=\hat{\mathbf{x}}$, $\hat{\mathbf{x}}_2=\hat{\mathbf{y}}$ and $\hat{\mathbf{x}}_3=\hat{\mathbf{z}}$.

Using ε_{ijk} we can write this in the compact form

$$\mathbf{c}=\sum_{i,j,k}\varepsilon_{ijk}\hat{\mathbf{x}}_i a_j b_k \tag{4}$$

as can be verified by expanding the sum and comparing with 3.

The Levi-Civita symbol can be used to write a completely antisymmetric wave function for a set of three fermions. Suppose the wave function for a single fermion in state n with coordinate x_a is $U_n(x_a)$ (where both n and a

can take values 1, 2 or 3). Then a completely antisymmetric wave function is

$$\psi_A(x_1,x_2,x_3) = \frac{1}{\sqrt{6}}\sum_{i,j,k}\varepsilon_{ijk}U_i(x_1)U_j(x_2)U_k(x_3) \tag{5}$$

The factor of $\frac{1}{\sqrt{6}}$ is for normalization and assumes that the U_n are all normalized wave functions.

Swapping the locations x_1 and x_2, for example, is equivalent to swapping i and j in the sum, which produces the negative of the original sum. That is

$$\psi_A(x_2,x_1,x_3) = \frac{1}{\sqrt{6}}\sum_{i,j,k}\varepsilon_{ijk}U_i(x_2)U_j(x_1)U_k(x_3) \tag{6}$$

$$= \frac{1}{\sqrt{6}}\sum_{i,j,k}\varepsilon_{jik}U_i(x_1)U_j(x_2)U_k(x_3) \tag{7}$$

$$= -\frac{1}{\sqrt{6}}\sum_{i,j,k}\varepsilon_{ijk}U_i(x_1)U_j(x_2)U_k(x_3) \tag{8}$$

$$= -\psi_A(x_1,x_2,x_3) \tag{9}$$

The same argument applies to swapping the other pairs of locations.

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