

## ANGULAR MOMENTUM IN THREE DIMENSIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.4.2.

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We can now generalize our treatment of rotation, originally studied in two dimensions, to three dimensions. We'll view a 3-d rotation as a combination of rotations about the  $x$ ,  $y$  and  $z$  axes, each of which can be represented by a  $3 \times 3$  matrix. These matrices are as follows:

$$(1) \quad R(\theta \hat{\mathbf{x}}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$(2) \quad R(\theta \hat{\mathbf{y}}) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$(3) \quad R(\theta \hat{\mathbf{z}}) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We're interested in infinitesimal rotations, for which we retain terms up to first order in the rotation angle  $\varepsilon_i$ , so that  $\cos \varepsilon_i = 1$  and  $\sin \varepsilon_i = \varepsilon_i$ . This gives the infinitesimal rotation matrices as

$$(4) \quad R(\varepsilon_x \hat{\mathbf{x}}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon_x \\ 0 & \varepsilon_x & 1 \end{bmatrix}$$

$$(5) \quad R(\varepsilon_y \hat{\mathbf{y}}) = \begin{bmatrix} 1 & 0 & \varepsilon_y \\ 0 & 1 & 0 \\ -\varepsilon_y & 0 & 1 \end{bmatrix}$$

$$(6) \quad R(\varepsilon_z \hat{\mathbf{z}}) = \begin{bmatrix} 1 & -\varepsilon_z & 0 \\ \varepsilon_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We now consider the series of rotations as follows: first, a rotation by  $\varepsilon_x \hat{\mathbf{x}}$ , then by  $\varepsilon_y \hat{\mathbf{y}}$ , then by  $-\varepsilon_x \hat{\mathbf{x}}$  and finally by  $-\varepsilon_y \hat{\mathbf{y}}$ . Because the various rotations don't commute, we don't end up back where we started. We can calculate the matrix products to find the final rotation.

(7)

$$R = R(-\varepsilon_y \hat{\mathbf{y}}) R(-\varepsilon_x \hat{\mathbf{x}}) R(\varepsilon_y \hat{\mathbf{y}}) R(\varepsilon_x \hat{\mathbf{x}})$$

(8)

$$= \begin{bmatrix} 1 & 0 & -\varepsilon_y \\ 0 & 1 & 0 \\ \varepsilon_y & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \varepsilon_x \\ 0 & -\varepsilon_x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \varepsilon_y \\ 0 & 1 & 0 \\ -\varepsilon_y & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon_x \\ 0 & \varepsilon_x & 1 \end{bmatrix}$$

(9)

$$= \begin{bmatrix} 1 & \varepsilon_x \varepsilon_y & -\varepsilon_y \\ 0 & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{bmatrix} \begin{bmatrix} 1 & \varepsilon_x \varepsilon_y & \varepsilon_y \\ 0 & 1 & -\varepsilon_x \\ -\varepsilon_y & \varepsilon_x & 1 \end{bmatrix}$$

(10)

$$= \begin{bmatrix} 1 + \varepsilon_y^2 & \varepsilon_x \varepsilon_y & -\varepsilon_x^2 \varepsilon_y \\ -\varepsilon_x \varepsilon_y & 1 + \varepsilon_x^2 & 0 \\ 0 & \varepsilon_x \varepsilon_y^2 & 1 + \varepsilon_x^2 + \varepsilon_y^2 \end{bmatrix}$$

To get the third line, we multiplied the first two matrices in the second line, and the last two matrices in the second line. In the final result, we can discard terms containing  $\varepsilon_x^2$  or  $\varepsilon_y^2$  to get

$$(11) \quad R = \begin{bmatrix} 1 & \varepsilon_x \varepsilon_y & 0 \\ -\varepsilon_x \varepsilon_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R(-\varepsilon_x \varepsilon_y \hat{\mathbf{z}})$$

Thus the result of the four rotations about the  $x$  and  $y$  axes is a single rotation about the  $z$  axis.

To convert this to quantum operators, we define the operator  $U[R]$  by comparison with the procedure we used for 2-d rotations. That is, the operator  $U$  is given by the corresponding angular momentum operator  $L_x$ ,  $L_y$  or  $L_z$  as

$$(12) \quad U [R(\boldsymbol{\varepsilon}_x \hat{\mathbf{x}})] = I - \frac{i\boldsymbol{\varepsilon}_x L_x}{\hbar}$$

$$(13) \quad U [R(\boldsymbol{\varepsilon}_y \hat{\mathbf{y}})] = I - \frac{i\boldsymbol{\varepsilon}_y L_y}{\hbar}$$

$$(14) \quad U [R(\boldsymbol{\varepsilon}_z \hat{\mathbf{z}})] = I - \frac{i\boldsymbol{\varepsilon}_z L_z}{\hbar}$$

By comparing 7 and 11 we thus require these  $U$  operators to satisfy

$$(15) \quad U [R(-\boldsymbol{\varepsilon}_y \hat{\mathbf{y}})] U [R(-\boldsymbol{\varepsilon}_x \hat{\mathbf{x}})] U [R(\boldsymbol{\varepsilon}_y \hat{\mathbf{y}})] U [R(\boldsymbol{\varepsilon}_x \hat{\mathbf{x}})] = U [R(-\boldsymbol{\varepsilon}_x \boldsymbol{\varepsilon}_y \hat{\mathbf{z}})]$$

We can get the commutation relation  $[L_x, L_y]$  by matching coefficients of  $\boldsymbol{\varepsilon}_x \boldsymbol{\varepsilon}_y$  on each side of this equation. On the RHS, the coefficient is  $\frac{iL_z}{\hbar}$ . On the LHS, we can pick out the terms involving  $\boldsymbol{\varepsilon}_x \boldsymbol{\varepsilon}_y$  to get

$$(16) \quad -\frac{1}{\hbar^2} (L_y L_x - L_x L_y - L_x L_y + L_y L_x) = \frac{1}{\hbar^2} [L_x, L_y]$$

The first term on the LHS comes from the  $\boldsymbol{\varepsilon}_x$  term in the first  $U$  in 15 multiplied by the  $\boldsymbol{\varepsilon}_y$  term in the second  $U$  (with the  $I$  term in the other two  $U$ s); the second term on the LHS comes from the  $\boldsymbol{\varepsilon}_x$  term in the first  $U$  in 15 multiplied by the  $\boldsymbol{\varepsilon}_y$  term in the fourth  $U$ , and so on.

Matching the two sides, we get

$$(17) \quad [L_x, L_y] = i\hbar L_z$$

By comparison with the classical definitions of the three components of  $\mathbf{L}$ , we can write the quantum operators in terms of position and momentum operators as

$$(18) \quad L_x = YP_z - ZP_y$$

$$(19) \quad L_y = ZP_x - XP_z$$

$$(20) \quad L_z = XP_y - YP_x$$

From the commutators of position and momentum  $[X, P_x] = i\hbar$  and so on, we can verify 17 from these relations as well.

$$\begin{aligned}
(21) \quad [L_x, L_y] &= [Y P_z - Z P_y, Z P_x - X P_z] \\
(22) \quad &= [Y P_z, Z P_x - X P_z] - [Z P_y, Z P_x - X P_z] \\
(23) \quad &= -i\hbar Y P_x + i\hbar P_y X \\
(24) \quad &= i\hbar (X P_y - Y P_x) \\
(25) \quad &= i\hbar L_z
\end{aligned}$$

The third line follows because  $[Y P_z, X P_z] = [Z P_y, Z P_x] = 0$ . The other two commutation relations follow by cyclic permutation of  $x$ ,  $y$  and  $z$ :

$$\begin{aligned}
(26) \quad [L_y, L_z] &= i\hbar L_x \\
(27) \quad [L_z, L_x] &= i\hbar L_y
\end{aligned}$$

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