ANGULAR MOMENTUM IN THREE DIMENSIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.4.2.

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We can now generalize our treatment of rotation, originally studied in two dimensions, to three dimensions. We'll view a 3-d rotation as a combination of rotations about the x, y and z axes, each of which can be represented by a 3×3 matrix. These matrices are as follows:

$$R(\theta\hat{\mathbf{x}}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{0.1}$$

$$R(\theta\hat{\mathbf{y}}) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{0.2}$$

$$R(\theta\hat{\mathbf{z}}) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{0.3}$$

We're interested in infinitesimal rotations, for which we retain terms up to first order in the rotation angle ε_i, so that $\cos\varepsilon_i = 1$ and $\sin\varepsilon_i = \varepsilon_i$. This gives the infinitesimal rotation matrices as

$$R(\varepsilon_x\hat{\mathbf{x}}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon_x \\ 0 & \varepsilon_x & 1 \end{bmatrix} \tag{0.4}$$

$$R(\varepsilon_y\hat{\mathbf{y}}) = \begin{bmatrix} 1 & 0 & \varepsilon_y \\ 0 & 1 & 0 \\ -\varepsilon_y & 0 & 1 \end{bmatrix} \tag{0.5}$$

$$R(\varepsilon_z\hat{\mathbf{z}}) = \begin{bmatrix} 1 & -\varepsilon_z & 0 \\ \varepsilon_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{0.6}$$

We now consider the series of rotations as follows: first, a rotation by $\varepsilon_x\hat{\mathbf{x}}$, then by $\varepsilon_y\hat{\mathbf{y}}$, then by $-\varepsilon_x\hat{\mathbf{x}}$ and finally by $-\varepsilon_y\hat{\mathbf{y}}$. Because the various rotations don't commute, we don't end up back where we started. We can calculate the matrix products to find the final rotation.

(0.7)

$$R = R(-\varepsilon_y\hat{\mathbf{y}})\,R(-\varepsilon_x\hat{\mathbf{x}})\,R(\varepsilon_y\hat{\mathbf{y}})\,R(\varepsilon_x\hat{\mathbf{x}})$$

(0.8)

$$= \begin{bmatrix} 1 & 0 & -\varepsilon_y \\ 0 & 1 & 0 \\ \varepsilon_y & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \varepsilon_x \\ 0 & -\varepsilon_x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \varepsilon_y \\ 0 & 1 & 0 \\ -\varepsilon_y & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon_x \\ 0 & \varepsilon_x & 1 \end{bmatrix}$$

(0.9)

$$= \begin{bmatrix} 1 & \varepsilon_x\varepsilon_y & -\varepsilon_y \\ 0 & 1 & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & 1 \end{bmatrix} \begin{bmatrix} 1 & \varepsilon_x\varepsilon_y & \varepsilon_y \\ 0 & 1 & -\varepsilon_x \\ -\varepsilon_y & \varepsilon_x & 1 \end{bmatrix}$$

(0.10)

$$= \begin{bmatrix} 1+\varepsilon_y^2 & \varepsilon_x\varepsilon_y & -\varepsilon_x^2\varepsilon_y \\ -\varepsilon_x\varepsilon_y & 1+\varepsilon_x^2 & 0 \\ 0 & \varepsilon_x\varepsilon_y^2 & 1+\varepsilon_x^2+\varepsilon_y^2 \end{bmatrix}$$

To get the third line, we multiplied the first two matrices in the second line, and the last two matrices in the second line. In the final result, we can discard terms containing ε_x^2 or ε_y^2 to get

(0.11)
$$R = \begin{bmatrix} 1 & \varepsilon_x\varepsilon_y & 0 \\ -\varepsilon_x\varepsilon_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R(-\varepsilon_x\varepsilon_y\hat{\mathbf{z}})$$

Thus the result of the four rotations about the x and y axes is a single rotation about the z axis.

To convert this to quantum operators, we define the operator $U[R]$ by comparison with the procedure we used for 2-d rotations. That is, the operator U is given by the corresponding angular momentum operator L_x, L_y or L_z as

$$U\left[R\left(\varepsilon_x\hat{\mathbf{x}}\right)\right] = I - \frac{i\varepsilon_x L_x}{\hbar} \tag{0.12}$$

$$U\left[R\left(\varepsilon_y\hat{\mathbf{y}}\right)\right] = I - \frac{i\varepsilon_y L_y}{\hbar} \tag{0.13}$$

$$U\left[R\left(\varepsilon_z\hat{\mathbf{z}}\right)\right] = I - \frac{i\varepsilon_z L_z}{\hbar} \tag{0.14}$$

By comparing 0.7 and 0.11 we thus require these U operators to satisfy

$$U\left[R\left(-\varepsilon_y\hat{\mathbf{y}}\right)\right]U\left[R\left(-\varepsilon_x\hat{\mathbf{x}}\right)\right]U\left[R\left(\varepsilon_y\hat{\mathbf{y}}\right)\right]U\left[R\left(\varepsilon_x\hat{\mathbf{x}}\right)\right] = U\left[R\left(-\varepsilon_x\varepsilon_y\hat{\mathbf{z}}\right)\right] \tag{0.15}$$

We can get the commutation relation $[L_x, L_y]$ by matching coefficients of $\varepsilon_x\varepsilon_y$ on each side of this equation. On the RHS, the coefficient is $\frac{iL_z}{\hbar}$. On the LHS, we can pick out the terms involving $\varepsilon_x\varepsilon_y$ to get

$$-\frac{1}{\hbar^2}\left(L_yL_x - L_yL_x - L_xL_y + L_yL_x\right) = \frac{1}{\hbar^2}\left[L_x, L_y\right] \tag{0.16}$$

The first term on the LHS comes from the ε_x term in the first U in 0.15 multiplied by the ε_y term in the second U (with the I term in the other two Us); the second term on the LHS comes from the ε_x term in the first U in 0.15 multiplied by the ε_y term in the fourth U, and so on.

Matching the two sides, we get

$$\left[L_x, L_y\right] = i\hbar L_z \tag{0.17}$$

By comparison with the classical definitions of the three components of $\mathbf{L}$, we can write the quantum operators in terms of position and momentum operators as

$$L_x = YP_z - ZP_y \tag{0.18}$$

$$L_y = ZP_x - XP_z \tag{0.19}$$

$$L_z = XP_y - YP_x \tag{0.20}$$

From the commutators of position and momentum $[X, P_x] = i\hbar$ and so on, we can verify 0.17 from these relations as well.

$$[L_x, L_y] = [YP_z - ZP_y, ZP_x - XP_z] \tag{0.21}$$

$$= [YP_z, ZP_x - XP_z] - [ZP_y, ZP_x - XP_z] \tag{0.22}$$

$$= -i\hbar YP_x + i\hbar P_y X \tag{0.23}$$

$$= i\hbar\,(XP_y - YP_x) \tag{0.24}$$

$$= i\hbar L_z \tag{0.25}$$

The third line follows because $[YP_z, XP_z] = [ZP_y, ZP_x] = 0$. The other two commutation relations follow by cyclic permutation of x, y and z:

$$[L_y, L_z] = i\hbar L_x \tag{0.26}$$

$$[L_z, L_x] = i\hbar L_y \tag{0.27}$$

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