

FINITE ROTATIONS ABOUT AN ARBITRARY AXIS IN THREE DIMENSIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.4.3.

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The operators for an infinitesimal rotation in 3-d are

$$U [R(\epsilon_x \hat{\mathbf{x}})] = I - \frac{i\epsilon_x L_x}{\hbar} \quad (1)$$

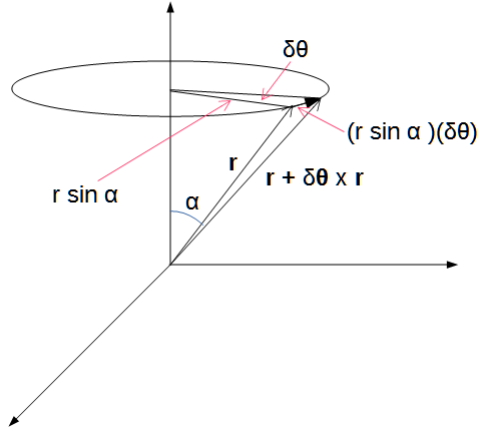
$$U [R(\epsilon_y \hat{\mathbf{y}})] = I - \frac{i\epsilon_y L_y}{\hbar} \quad (2)$$

$$U [R(\epsilon_z \hat{\mathbf{z}})] = I - \frac{i\epsilon_z L_z}{\hbar} \quad (3)$$

If we have a finite (larger than infinitesimal) rotation about one of the coordinate axes, we can create the operator by dividing up the finite rotation angle θ into N small increments and take the limit as $N \rightarrow \infty$, just as we did with finite translations. For example, for a finite rotation about the x axis, we have

$$U [R(\theta \hat{\mathbf{x}})] = \lim_{N \rightarrow \infty} \left(I - \frac{i\theta L_x}{N\hbar} \right)^N = e^{-i\theta L_x/\hbar} \quad (4)$$

What if we have a finite rotation about some arbitrarily directed axis? Suppose we have a vector \mathbf{r} as shown in the figure:



The vector \mathbf{r} makes an angle α with the z axis, and we wish to rotate \mathbf{r} about the z axis by an angle $\delta\theta$. Note that this argument is completely general, since if the axis of rotation is not the z axis, we can rotate the entire coordinate system so that the axis of rotation *is* the z axis. The generality enters through the fact that we're keeping the angle α arbitrary.

The rotation by $\delta\theta\hat{\mathbf{z}} \equiv \delta\theta$ shifts the tip of \mathbf{r} along the circle shown by a distance $(r \sin \alpha) \delta\theta$ in a counterclockwise direction (looking down the z axis). This shift is in a direction that is perpendicular to both $\hat{\mathbf{z}}$ and \mathbf{r} , so the little vector representing the shift in \mathbf{r} is

$$\delta\mathbf{r} = (\delta\theta) \times \mathbf{r} \quad (5)$$

Thus under the rotation $\delta\theta$, a vector transforms as

$$\mathbf{r} \rightarrow \mathbf{r} + (\delta\theta) \times \mathbf{r} \quad (6)$$

Just as with translations, if we rotate the coordinate system by an amount $\delta\theta$, this is equivalent to rotating the wave function $\psi(\mathbf{r})$ by the same angle, but in the opposite direction, so we require

$$\psi(\mathbf{r}) \rightarrow \psi(\mathbf{r} - (\delta\theta) \times \mathbf{r}) \quad (7)$$

A first order Taylor expansion of the quantity on the RHS gives

$$\psi(\mathbf{r} - (\delta\theta) \times \mathbf{r}) = \psi(\mathbf{r}) - (\delta\theta \times \mathbf{r}) \cdot \nabla \psi \quad (8)$$

The operator generating this rotation will have the form (in analogy with the forms for the coordinate axes above):

$$U[R(\delta\theta)] = I - \frac{i\delta\theta}{\hbar} L_{\hat{\theta}} \quad (9)$$

where $L_{\hat{\theta}}$ is an angular momentum operator to be determined.

Writing out the RHS of 8, we have

$$\psi(\mathbf{r}) - (\delta\boldsymbol{\theta} \times \mathbf{r}) \cdot \nabla\psi = \psi(\mathbf{r}) - (\delta\theta_{yz} - \delta\theta_{zy}) \frac{\partial\psi}{\partial x} + (\delta\theta_{xz} - \delta\theta_{zx}) \frac{\partial\psi}{\partial y} - (\delta\theta_{xy} - \delta\theta_{yx}) \frac{\partial\psi}{\partial z} \quad (10)$$

$$= \psi(\mathbf{r}) - \delta\theta_x \left(y \frac{\partial\psi}{\partial z} - z \frac{\partial\psi}{\partial y} \right) - \delta\theta_y \left(z \frac{\partial\psi}{\partial x} - x \frac{\partial\psi}{\partial z} \right) - \delta\theta_z \left(x \frac{\partial\psi}{\partial y} - y \frac{\partial\psi}{\partial x} \right) \quad (11)$$

$$= \psi(\mathbf{r}) - \delta\boldsymbol{\theta} \cdot \frac{i}{\hbar} \mathbf{r} \times \mathbf{p} \psi \quad (12)$$

$$= \psi(\mathbf{r}) - \frac{i}{\hbar} \delta\boldsymbol{\theta} \cdot \mathbf{L} \psi \quad (13)$$

$$= U[R(\delta\boldsymbol{\theta})] \psi \quad (14)$$

Comparing this with 9, we see that

$$L_{\hat{\boldsymbol{\theta}}} = \hat{\boldsymbol{\theta}} \cdot \mathbf{L} \quad (15)$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector along the axis of rotation. Since all rotations about the same axis commute, we can use the same procedure as above to generate a finite rotation $\boldsymbol{\theta}$ about an arbitrary axis and get

$$U[R(\boldsymbol{\theta})] = e^{-i\boldsymbol{\theta} \cdot \mathbf{L}/\hbar} \quad (16)$$

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