FINITE ROTATIONS ABOUT AN ARBITRARY AXIS IN THREE DIMENSIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.4.3.

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The operators for an infinitesimal rotation in 3-d are

$$U\left[R\left(\varepsilon_x\hat{\mathbf{x}}\right)\right] = I-\frac{i\varepsilon_x L_x}{\hbar} \tag{1}$$

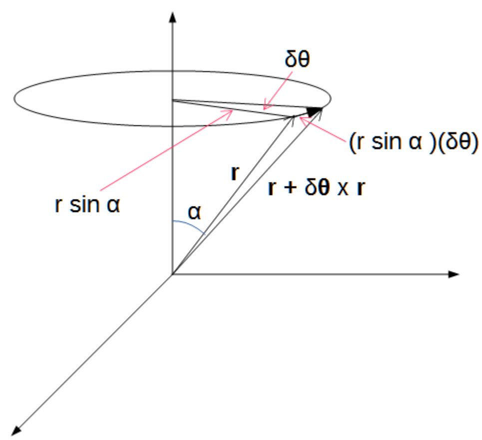
$$U\left[R\left(\varepsilon_y\hat{\mathbf{y}}\right)\right] = I-\frac{i\varepsilon_y L_y}{\hbar} \tag{2}$$

$$U\left[R\left(\varepsilon_z\hat{\mathbf{z}}\right)\right] = I-\frac{i\varepsilon_z L_z}{\hbar} \tag{3}$$

If we have a finite (larger than infinitesimal) rotation about one of the coordinate axes, we can create the operator by dividing up the finite rotation angle θ into N small increments and take the limit as $N\to\infty$, just as we did with finite translations. For example, for a finite rotation about the x axis, we have

$$U\left[R\left(\theta\hat{\mathbf{x}}\right)\right]=\lim_{N\to\infty}\left(I-\frac{i\theta L_x}{N\hbar}\right)^N=e^{-i\theta L_x/\hbar} \tag{4}$$

What if we have a finite rotation about some arbitrarily directed axis? Suppose we have a vector $\mathbf{r}$ as shown in the figure:



The vector $\mathbf{r}$ makes an angle α with the z axis, and we wish to rotate $\mathbf{r}$ about the z axis by an angle $\delta\theta$. Note that this argument is completely general, since if the axis of rotation is not the z axis, we can rotate the entire coordinate system so that the axis of rotation *is* the z axis. The generality enters through the fact that we're keeping the angle α arbitrary.

The rotation by $\delta\theta\hat{\mathbf{z}} \equiv \delta\boldsymbol{\theta}$ shifts the tip of $\mathbf{r}$ along the circle shown by a distance $(r\sin\alpha)\,\delta\theta$ in a counterclockwise direction (looking down the z axis). This shift is in a direction that is perpendicular to both $\hat{\mathbf{z}}$ and $\mathbf{r}$, so the little vector representing the shift in $\mathbf{r}$ is

$$\delta\mathbf{r} = (\delta\boldsymbol{\theta}) \times \mathbf{r} \tag{5}$$

Thus under the rotation $\delta\boldsymbol{\theta}$, a vector transforms as

$$\mathbf{r} \to \mathbf{r} + (\delta\boldsymbol{\theta}) \times \mathbf{r} \tag{6}$$

Just as with translations, if we rotate the coordinate system by an amount $\delta\boldsymbol{\theta}$, this is equivalent to rotating the wave function $\psi(\mathbf{r})$ by the same angle, but in the opposite direction, so we require

$$\psi(\mathbf{r}) \to \psi(\mathbf{r} - (\delta\boldsymbol{\theta}) \times \mathbf{r}) \tag{7}$$

A first order Taylor expansion of the quantity on the RHS gives

$$\psi(\mathbf{r} - (\delta\boldsymbol{\theta}) \times \mathbf{r}) = \psi(\mathbf{r}) - (\delta\boldsymbol{\theta} \times \mathbf{r}) \cdot \nabla\psi \tag{8}$$

The operator generating this rotation will have the form (in analogy with the forms for the coordinate axes above):

$$U[R(\delta\theta)] = I - \frac{i\delta\theta}{\hbar} L_{\hat{\theta}} \tag{9}$$

where $L_{\hat{\theta}}$ is an angular momentum operator to be determined.
Writing out the RHS of 8, we have

$$\psi(\mathbf{r}) - (\delta\theta \times \mathbf{r}) \cdot \nabla\psi = \psi(\mathbf{r}) - (\delta\theta_y z - \delta\theta_z y)\frac{\partial\psi}{\partial x} + (\delta\theta_x z - \delta\theta_z x)\frac{\partial\psi}{\partial y} - (\delta\theta_x y - \delta\theta_y x)\frac{\partial\psi}{\partial z} \tag{10}$$

$$= \psi(\mathbf{r}) - \delta\theta_x \left(y\frac{\partial\psi}{\partial z} - z\frac{\partial\psi}{\partial y} \right) - \delta\theta_y \left(z\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial z} \right) - \delta\theta_z \left(x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x} \right) \tag{11}$$

$$= \psi(\mathbf{r}) - \delta\theta \cdot \frac{i}{\hbar}\mathbf{r} \times \mathbf{p}\psi \tag{12}$$

$$= \psi(\mathbf{r}) - \frac{i}{\hbar}\delta\theta \cdot \mathbf{L}\psi \tag{13}$$

$$= U[R(\delta\theta)]\,\psi \tag{14}$$

Comparing this with 9, we see that

$$L_{\hat{\theta}} = \hat{\theta} \cdot \mathbf{L} \tag{15}$$

where $\hat{\theta}$ is the unit vector along the axis of rotation. Since all rotations about the same axis commute, we can use the same procedure as above to generate a finite rotation θ about an arbitrary axis and get

$$U[R(\theta)] = e^{-i\theta\cdot\mathbf{L}/\hbar} \tag{16}$$

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