# VECTOR OPERATORS; TRANSFORMATION UNDER ROTATION

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.4.4.

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A vector operator $\mathbf{V}$ is defined as an operator whose components transform under rotation according to

$$U^{\dagger}[R]V_iU[R]=\sum_j R_{ij}V_j \tag{1}$$

where $R$ is the rotation matrix in either 2 or 3 dimensions. We've seen that, for an infinitesimal rotation about an arbitrary axis $\delta\theta$, a vector transforms like

$$\mathbf{V}\rightarrow\mathbf{V}+\delta\theta\times\mathbf{V} \tag{2}$$

This can be written more compactly using the Levi-Civita tensor, since component $i$ of a cross product is

$$(\delta\theta\times\mathbf{V})_i=\sum_{j,k}\varepsilon_{ijk}(\delta\theta)_jV_k \tag{3}$$

We get

$$\sum_j R_{ij}V_j=V_i+\sum_{j,k}\varepsilon_{ijk}(\delta\theta)_jV_k \tag{4}$$

The operator $U[R]$ is given by

$$U[R(\delta\theta)]=I-\frac{i}{\hbar}\delta\theta\cdot\mathbf{L} \tag{5}$$

where $\mathbf{L}$ is the angular momentum. Plugging this into 1, we have, to first order in $\delta\theta$ (remembering that the components of $\mathbf{L}$ do not commute with each other and, in general also do not commute with the components of $\mathbf{V}$):

(6)
$$\left(I+\frac{i}{\hbar}\delta\theta\cdot\mathbf{L}\right)V_i\left(I-\frac{i}{\hbar}\delta\theta\cdot\mathbf{L}\right)=V_i+\frac{i}{\hbar}\sum_j\left(\delta\theta_jL_j\right)V_i-\frac{i}{\hbar}V_i\sum_j\left(\delta\theta_jL_j\right)$$

(7)
$$=V_i+\frac{i}{\hbar}\sum_j\delta\theta_j\left[L_j,V_i\right]$$

Setting this equal to the RHS of 4 we have, equating coefficients of $\delta\theta_j$:

(8)
$$\frac{i}{\hbar}\left[L_j,V_i\right] = \sum_k\varepsilon_{ijk}V_k$$

(9)
$$\left[V_i,L_j\right] = i\hbar\sum_k\varepsilon_{ijk}V_k$$

With $\mathbf{V}=\mathbf{L}$, we regain the commutation relations for the components of angular momentum

(10)
$$\left[L_x,L_y\right] = i\hbar L_z$$

(11)
$$\left[L_y,L_z\right] = i\hbar L_x$$

(12)
$$\left[L_z,L_x\right] = i\hbar L_y$$

By the way, it is possible to write these commutation relations in the compact form

(13)
$$\mathbf{L}\times\mathbf{L}=i\hbar\mathbf{L}$$

This looks wrong if you're used to the standard definition of the cross product for vectors whose components are ordinary numbers, since for such a vector $\mathbf{a}$, we always have $\mathbf{a}\times\mathbf{a}=0$. However, if the components of the vector are *operators* that don't commute, then the result is not zero, as we can see:

(14)
$$\left(\mathbf{L}\times\mathbf{L}\right)_i = \sum_{j,k}\varepsilon_{ijk}L_jL_k$$

If $i=x$, for example, then the sum on the RHS gives

(15)
$$\left(\mathbf{L}\times\mathbf{L}\right)_x = \sum_{j,k}\varepsilon_{xjk}L_jL_k$$

(16)
$$= L_yL_z-L_zL_y$$

(17)
$$= \left[L_y,L_z\right]$$

From 13, this gives

$$[L_y, L_z] = i\hbar L_x \tag{18}$$

## PINGBACKS

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