A vector operator \( \mathbf{V} \) is defined as an operator whose components transform under rotation according to

\[
U^\dagger [R] V_i U [R] = \sum_j R_{ij} V_j
\]

(1)

where \( R \) is the rotation matrix in either 2 or 3 dimensions. We’ve seen that, for an infinitesimal rotation about an arbitrary axis \( \delta \theta \), a vector transforms like

\[
\mathbf{V} \rightarrow \mathbf{V} + \delta \theta \times \mathbf{V}
\]

(2)

This can be written more compactly using the Levi-Civita tensor, since component \( i \) of a cross product is

\[
(\delta \theta \times \mathbf{V})_i = \sum_{j,k} \varepsilon_{ijk} (\delta \theta)_j V_k
\]

(3)

We get

\[
\sum_j R_{ij} V_j = V_i + \sum_{j,k} \varepsilon_{ijk} (\delta \theta)_j V_k
\]

(4)

The operator \( U [R] \) is given by

\[
U [R (\delta \theta)] = I - \frac{i}{\hbar} \bar{\delta \theta} \cdot \mathbf{L}
\]

(5)

where \( \mathbf{L} \) is the angular momentum. Plugging this into (1) we have, to first order in \( \delta \theta \) (remembering that the components of \( \mathbf{L} \) do not commute with each other and, in general also do not commute with the components of \( \mathbf{V} \)): 
\( \left( I + \frac{i}{\hbar} \delta \theta \cdot \mathbf{L} \right) V_i \left( I - \frac{i}{\hbar} \delta \theta \cdot \mathbf{L} \right) = V_i + \frac{i}{\hbar} \sum_j (\delta \theta_j L_j) V_i - \frac{i}{\hbar} V_i \sum_j (\delta \theta_j L_j) \)

\[ (6) \]

\[ (7) \]

Setting this equal to the RHS of 4 we have, equating coefficients of \( \delta \theta_j \):

\[ \frac{i}{\hbar} [L_j, V_i] = \sum_k \epsilon_{ijk} V_k \quad (8) \]

\[ [V_i, L_j] = i\hbar \sum_k \epsilon_{ijk} V_k \quad (9) \]

With \( \mathbf{V} = \mathbf{L} \), we regain the **commutation relations** for the components of angular momentum

\[ [L_x, L_y] = i\hbar L_z \quad (10) \]

\[ [L_y, L_z] = i\hbar L_x \quad (11) \]

\[ [L_z, L_x] = i\hbar L_y \quad (12) \]

By the way, it is possible to write these commutation relations in the compact form

\[ \mathbf{L} \times \mathbf{L} = i\hbar \mathbf{L} \quad (13) \]

This looks wrong if you’re used to the standard definition of the cross product for vectors whose components are ordinary numbers, since for such a vector \( \mathbf{a} \), we always have \( \mathbf{a} \times \mathbf{a} = 0 \). However, if the components of the vector are **operators** that don’t commute, then the result is not zero, as we can see:

\[ (\mathbf{L} \times \mathbf{L})_i = \sum_{j,k} \epsilon_{ijk} L_j L_k \quad (14) \]

If \( i = x \), for example, then the sum on the RHS gives

\[ (\mathbf{L} \times \mathbf{L})_x = \sum_{j,k} \epsilon_{xjk} L_j L_k \quad (15) \]

\[ = L_y L_z - L_z L_y \quad (16) \]

\[ = [L_y, L_z] \quad (17) \]

From 13, this gives
\[ [L_y, L_z] = i\hbar L_x \quad (18) \]

**Pingbacks**

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