

VECTOR OPERATORS; TRANSFORMATION UNDER ROTATION

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.4.4.

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A vector operator \mathbf{V} is defined as an operator whose components transform under rotation according to

$$(0.1) \quad U^\dagger [R] V_i U [R] = \sum_j R_{ij} V_j$$

where R is the rotation matrix in either 2 or 3 dimensions. We've seen that, for an infinitesimal rotation about an arbitrary axis $\delta\theta$, a vector transforms like

$$(0.2) \quad \mathbf{V} \rightarrow \mathbf{V} + \delta\theta \times \mathbf{V}$$

This can be written more compactly using the Levi-Civita tensor, since component i of a cross product is

$$(0.3) \quad (\delta\theta \times \mathbf{V})_i = \sum_{j,k} \epsilon_{ijk} (\delta\theta)_j V_k$$

We get

$$(0.4) \quad \sum_j R_{ij} V_j = V_i + \sum_{j,k} \epsilon_{ijk} (\delta\theta)_j V_k$$

The operator $U [R]$ is given by

$$(0.5) \quad U [R(\delta\theta)] = I - \frac{i}{\hbar} \delta\theta \cdot \mathbf{L}$$

where \mathbf{L} is the angular momentum. Plugging this into 0.1, we have, to first order in $\delta\theta$ (remembering that the components of \mathbf{L} do not commute

with each other and, in general also do not commute with the components of \mathbf{V}):

$$(0.6) \quad \left(I + \frac{i}{\hbar} \delta\theta \cdot \mathbf{L} \right) V_i \left(I - \frac{i}{\hbar} \delta\theta \cdot \mathbf{L} \right) = V_i + \frac{i}{\hbar} \sum_j (\delta\theta_j L_j) V_i - \frac{i}{\hbar} V_i \sum_j (\delta\theta_j L_j)$$

$$(0.7) \quad = V_i + \frac{i}{\hbar} \sum_j \delta\theta_j [L_j, V_i]$$

Setting this equal to the RHS of 0.4 we have, equating coefficients of $\delta\theta_j$:

$$(0.8) \quad \frac{i}{\hbar} [L_j, V_i] = \sum_k \varepsilon_{ijk} V_k$$

$$(0.9) \quad [V_i, L_j] = i\hbar \sum_k \varepsilon_{ijk} V_k$$

With $\mathbf{V} = \mathbf{L}$, we regain the commutation relations for the components of angular momentum

$$(0.10) \quad [L_x, L_y] = i\hbar L_z$$

$$(0.11) \quad [L_y, L_z] = i\hbar L_x$$

$$(0.12) \quad [L_z, L_x] = i\hbar L_y$$

By the way, it is possible to write these commutation relations in the compact form

$$(0.13) \quad \mathbf{L} \times \mathbf{L} = i\hbar \mathbf{L}$$

This looks wrong if you're used to the standard definition of the cross product for vectors whose components are ordinary numbers, since for such a vector \mathbf{a} , we always have $\mathbf{a} \times \mathbf{a} = 0$. However, if the components of the vector are *operators* that don't commute, then the result is not zero, as we can see:

$$(0.14) \quad (\mathbf{L} \times \mathbf{L})_i = \sum_{j,k} \varepsilon_{ijk} L_j L_k$$

If $i = x$, for example, then the sum on the RHS gives

$$(0.15) \quad (\mathbf{L} \times \mathbf{L})_x = \sum_{j,k} \varepsilon_{xjk} L_j L_k$$

$$(0.16) \quad = L_y L_z - L_z L_y$$

$$(0.17) \quad = [L_y, L_z]$$

From 0.13, this gives

$$(0.18) \quad [L_y, L_z] = i\hbar L_x$$

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