# ROTATION OF A VECTOR WAVE FUNCTION

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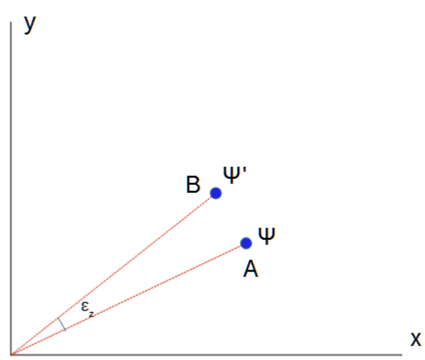
Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.1.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

We've seen that, for a rotation by an infinitesimal angle $\varepsilon_z$ about the $z$ axis, a scalar wave function transforms according to

$$\psi(x,y) \to \psi(x+\varepsilon_z y, y-\varepsilon_z x) \tag{1}$$

The meaning of this transformation can be seen in the figure:




The physical system represented by the wave function $\Psi$ is rigidly rotated by the angle $\varepsilon_z$, so that the value of $\Psi$ at point $A$ is now sitting over the point $B$. However, in the primed (rotated) coordinate system, the numerical value of the coordinates of the point $B$ in the figure are the same as the numerical values that the point $A$ had in the original, unrotated coordinates. That is

$$\left(x'_B, y'_B\right) = (x_A, y_A) \tag{2}$$

Just as $B$ is obtained from $A$ by rotating $A$ by $+\varepsilon_z$, we can obtain $A$ from $B$ by rotating by $-\varepsilon_z$. For any given point, the primed (rotated) and unprimed (unrotated) coordinates are related by (all relations are to first order in $\varepsilon_z$):

$$x' = x - y\varepsilon_z \tag{3}$$
$$y' = y + x\varepsilon_z \tag{4}$$

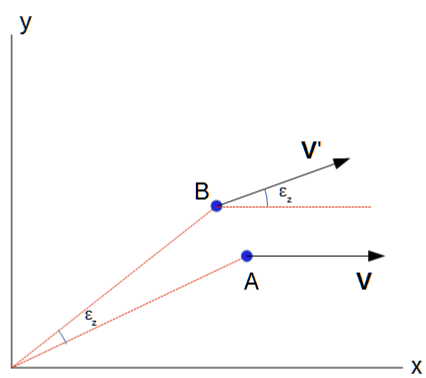
The inverse relations are obtained by a rotation by $-\varepsilon_z$:

$$x = x' + y'\varepsilon_z \tag{5}$$
$$y = y' - x'\varepsilon_z \tag{6}$$

After rotation, the values of $\Psi'$ are related to the values $\Psi$ before rotation by rotating through the angle $-\varepsilon_z$, so that

$$\Psi'(x,y) = \Psi(x + y\varepsilon_z, y - x\varepsilon_z) \tag{7}$$

Now suppose the wave function is a vector $\mathbf{V} = V_x\hat{\mathbf{x}} + V_y\hat{\mathbf{y}}$. The situation is as shown:




The initial unrotated vector $\mathbf{V}$ is the value of the wave function at point $A$ (and is entirely in the $x$ direction for convenience). After rotation, the vector gets moved to $B$ and is also rotated so that it now makes an angle $\varepsilon_z$ with the *original* $x$ axis. However, its direction is now along the $x'$ axis, which makes an angle of $\varepsilon_z$ with the original $x$ axis.

In this case, each component of $\mathbf{V}$ still gets transformed in the same way as the scalar function above, but the vector itself is also rotated. If the components $V_x$ and $V_y$ of the vector were constants, then the rotated vector is given by applying the 2-d rotation matrix

$$R = \begin{bmatrix} 1 & -\varepsilon_z \\ \varepsilon_z & 1 \end{bmatrix} \tag{8}$$

so we get $\mathbf{V}' = R\mathbf{V}$, or, in components:

$$V'_x = V_x - V_y\varepsilon_z \tag{9}$$
$$V'_y = V_y + V_x\varepsilon_z \tag{10}$$

If $V_x$ and $V_y$ vary from point to point, then we must apply the transformation 1 to each component, so that the overall transformation is

$$V'_x = V_x(x+\varepsilon_z y, y-\varepsilon_z x) - V_y(x+\varepsilon_z y, y-\varepsilon_z x)\varepsilon_z \tag{11}$$
$$V'_y = V_y(x+\varepsilon_z y, y-\varepsilon_z x) + V_x(x+\varepsilon_z y, y-\varepsilon_z x)\varepsilon_z \tag{12}$$

The operator that generates the transformation of a scalar function by an infinitesimal angle $\delta\theta$ is

$$U[R(\delta\theta)] = I - \frac{i}{\hbar}\delta\theta\cdot\mathbf{L} \tag{13}$$

In this case, the rotation is about the $z$ axis so

$$\delta\theta = \varepsilon_z\hat{\mathbf{z}} \tag{14}$$
$$\delta\theta\cdot\mathbf{L} = \varepsilon_z L_z \tag{15}$$

Thus we have

$$V_{x,y}(x+\varepsilon_z y, y-\varepsilon_z x) = \left(I - \frac{i}{\hbar}\varepsilon_z L_z\right)V_{x,y}(x,y) \tag{16}$$

Plugging this into 11 and keeping terms only up to order $\varepsilon_z$ we have

$$V'_x = \left(I - \frac{i}{\hbar}\varepsilon_z L_z\right)V_x - V_y\varepsilon_z \tag{17}$$
$$V'_y = \left(I - \frac{i}{\hbar}\varepsilon_z L_z\right)V_y + V_x\varepsilon_z \tag{18}$$

In matrix form, this is

$$\begin{bmatrix} V'_x \\ V'_y \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{i\varepsilon_z}{\hbar}\begin{bmatrix} L_z & 0 \\ 0 & L_z \end{bmatrix} - \varepsilon_z\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\begin{bmatrix} V_x \\ V_y \end{bmatrix} \tag{19}$$
$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{i\varepsilon_z}{\hbar}\begin{bmatrix} L_z & 0 \\ 0 & L_z \end{bmatrix} - \frac{i\varepsilon_z}{\hbar}\begin{bmatrix} 0 & -i\hbar \\ i\hbar & 0 \end{bmatrix}\right)\begin{bmatrix} V_x \\ V_y \end{bmatrix} \tag{20}$$
$$= \left(I - \frac{i\varepsilon_z}{\hbar}J_z\right)\begin{bmatrix} V_x \\ V_y \end{bmatrix} \tag{21}$$

This has the same form as 13, except that the angular momentum generator is now the sum of $L_z$ and the final matrix on the RHS above, which Shankar calls suggestively $S_z$, in anticipation of spin which at this stage he hasn't considered. That is,

$$J_z = L_z + S_z \tag{22}$$

$$= \left[\begin{array}{cc} L_z & 0 \\ 0 & L_z \end{array}\right] + \left[\begin{array}{cc} 0 & -i\hbar \\ i\hbar & 0 \end{array}\right] \tag{23}$$

The eigenvalues of the second matrix here are just $\pm\hbar$, so we haven't yet encountered half-integral values of angular momentum.

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