

TOTAL ANGULAR MOMENTUM - MATRIX ELEMENTS AND COMMUTATION RELATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.2.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In Shankar's Chapter 12 treatment of the eigenvalues of the angular momentum operators L^2 and L_z , he retraces much of what we've already covered as a result of working through Griffiths's book. He defines raising and lowering operators for angular momentum as

$$L_{\pm} \equiv L_x \pm iL_y \quad (1)$$

These operators can be used to discover the eigenvalues of L^2 to be $\ell(\ell+1)\hbar^2$, where $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ and the eigenvalues of L_z are $m\hbar$ where m ranges from $-\ell$ to $+\ell$ in integer steps. The eigenvalues of L_{\pm} can also be found to satisfy

$$L_{\pm} |\ell m\rangle = \hbar \sqrt{(\ell \mp m)(\ell \pm m + 1)} |\ell, m \pm 1\rangle \quad (2)$$

When dealing with vector wave functions (as opposed to scalar ones) in two dimensions, we found that a quantity J_z is the generator of infinitesimal rotations about the z axis, where

$$J_z = L_z + S_z \quad (3)$$

and the operator producing the rotation by ϵ_z is

$$U [R(\epsilon_z \hat{\mathbf{z}})] = I - \frac{i\epsilon_z}{\hbar} J_z \quad (4)$$

For a scalar wave function in three dimensions, we found that the properties of two successive rotations by ϵ_x about the x axis and ϵ_y about the y axis led to the commutations

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$$[L_x, L_y] = i\hbar L_z \quad (5)$$

$$[L_y, L_z] = i\hbar L_x \quad (6)$$

$$[L_z, L_x] = i\hbar L_y \quad (7)$$

For a vector wave function, the rotation is generated by J_i rather than L_i but because the effects of rotations are the same, the J_i must have the same commutation relations, so that

$$[J_x, J_y] = i\hbar J_z \quad (8)$$

$$[J_y, J_z] = i\hbar J_x \quad (9)$$

$$[J_z, J_x] = i\hbar J_y \quad (10)$$

We can do the same analysis on J as we did above with L to define the raising and lowering operators

$$J_{\pm} \equiv J_x \pm iJ_y \quad (11)$$

and get the same eigenvalue relations

$$J_{\pm} |jm\rangle = \hbar \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle \quad (12)$$

The three components of \mathbf{J} are then J_z and

$$J_x = \frac{1}{2} (J_+ + J_-) \quad (13)$$

$$J_y = \frac{1}{2i} (J_+ - J_-) \quad (14)$$

Using these three equations, we can generate the matrix elements of the components of \mathbf{J} in the orthonormal basis $|jm\rangle$ (that is, the basis consisting of eigenfunctions with total angular momentum number j and J_z number m). These matrix elements are

$$\langle j'm' | J_x | jm \rangle = \frac{1}{2} \langle j'm' | J_+ + J_- | jm \rangle \quad (15)$$

$$= \frac{\hbar}{2} \sqrt{(j-m)(j+m+1)} \langle j'm' | j, m+1 \rangle + \quad (16)$$

$$\frac{\hbar}{2} \sqrt{(j+m)(j-m+1)} \langle j'm' | j, m-1 \rangle \quad (17)$$

$$= \frac{\hbar}{2} \left[\sqrt{(j-m)(j+m+1)} \delta_{j'j} \delta_{m',m+1} + \sqrt{(j+m)(j-m+1)} \delta_{j'j} \delta_{m',m-1} \right] \quad (18)$$

$$\langle j'm' | J_y | jm \rangle = \frac{1}{2i} \langle j'm' | J_+ - J_- | jm \rangle \quad (19)$$

$$= \frac{\hbar}{2i} \sqrt{(j-m)(j+m+1)} \langle j'm' | j, m+1 \rangle - \quad (20)$$

$$\frac{\hbar}{2i} \sqrt{(j+m)(j-m+1)} \langle j'm' | j, m-1 \rangle \quad (21)$$

$$= \frac{\hbar}{2i} \left[\sqrt{(j-m)(j+m+1)} \delta_{j'j} \delta_{m',m+1} - \sqrt{(j+m)(j-m+1)} \delta_{j'j} \delta_{m',m-1} \right] \quad (22)$$

$$\langle j'm' | J_z | jm \rangle = m\hbar \delta_{j'j} \delta_{m',m} \quad (23)$$

The full matrix for each component J_i is actually infinite-dimensional, since j can be any half-integer from 0 up to infinity. However, the sub-matrix for each value of j is completely orthogonal to all other sub-matrices with different j values, so the complete matrix for each J_i is block-diagonal. Shankar gives the matrices for J_x and J_y up to $j = 1$ in his equations 12.5.23 and 12.5.24. This means that the commutation relations 9 should be obeyed for each set of sub-matrices corresponding to a particular j value.

For $j = \frac{1}{2}$ we have for the 3 sub-matrices (we can copy these from Shankar or use the above formulas to work them out). The values of m are $+\frac{1}{2}$ and $-\frac{1}{2}$ in that order, from top to bottom and left to right.

$$J_x^{(1/2)} = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (24)$$

$$J_y^{(1/2)} = \frac{\hbar}{2i} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (25)$$

$$= \frac{i\hbar}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (26)$$

$$[J_x^{(1/2)}, J_y^{(1/2)}] = \frac{i\hbar^2}{4} \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right) \quad (27)$$

$$\frac{i\hbar^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (28)$$

$$= i\hbar J_z^{(1/2)} \quad (29)$$

For $j = 1$ we have for the 3 sub-matrices

$$J_x^{(1)} = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (30)$$

$$J_y^{(1)} = \frac{i\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (31)$$

$$[J_x^{(1)}, J_y^{(1)}] = \frac{i\hbar^2}{2} \left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right) \quad (32)$$

$$= i\hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (33)$$

$$= i\hbar J_z^{(1)} \quad (34)$$

For $j = \frac{3}{2}$ we need to work out the matrices from the formulas above for the matrix elements. Ordering the values of $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}$ from left to right (columns) and top to bottom (rows), we get

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$$J_x^{(3/2)} = \frac{\hbar}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \quad (35)$$

$$J_y^{(3/2)} = \frac{\hbar}{2i} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{bmatrix} \quad (36)$$

$$= \frac{i\hbar}{2} \begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \quad (37)$$

$$[J_x^{(3/2)}, J_y^{(3/2)}] = \frac{i\hbar^2}{4} \left(\begin{bmatrix} 3 & 0 & -2\sqrt{3} & 0 \\ 0 & 1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & -1 & 0 \\ 0 & 2\sqrt{3} & 0 & -3 \end{bmatrix} - \begin{bmatrix} -3 & 0 & -2\sqrt{3} & 0 \\ 0 & -1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & 1 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{bmatrix} \right) \quad (38)$$

$$= i\hbar^2 \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix} \quad (39)$$

$$= i\hbar J_z^{(3/2)} \quad (40)$$

Thus the commutation relation $[J_x, J_y] = i\hbar J_z$ is satisfied for these three sets of sub-matrices.

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