TOTAL ANGULAR MOMENTUM - MATRIX ELEMENTS AND COMMUTATION RELATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.2.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In Shankar's Chapter 12 treatment of the eigenvalues of the angular momentum operators L^2 and L_z, he retraces much of what we've already covered as a result of working through Griffiths's book. He defines raising and lowering operators for angular momentum as

$$L_{\pm} \equiv L_x \pm iL_y \tag{1}$$

These operators can be used to discover the eigenvalues of L^2 to be $\ell(\ell+1)\hbar^2$, where $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and the eigenvalues of L_z are $m\hbar$ where m ranges from $-\ell$ to $+\ell$ in integer steps. The eigenvalues of $L_{\pm}$ can also be found to satisfy

$$L_{\pm}|\ell m\rangle = \hbar\sqrt{(\ell \mp m)(\ell \pm m+1)}\,|\ell, m\pm 1\rangle \tag{2}$$

When dealing with vector wave functions (as opposed to scalar ones) in two dimensions, we found that a quantity J_z is the generator of infinitesimal rotations about the z axis, where

$$J_z = L_z + S_z \tag{3}$$

and the operator producing the rotation by ε_z is

$$U[R(\varepsilon_z\hat{\mathbf{z}})] = I - \frac{i\varepsilon_z}{\hbar}J_z \tag{4}$$

For a scalar wave function in three dimensions, we found that the properties of two successive rotations by ε_x about the x axis and ε_y about the y axis led to the commutations

$$\begin{aligned}
[L_x, L_y] &= i\hbar L_z && (5)\\
[L_y, L_z] &= i\hbar L_x && (6)\\
[L_z, L_x] &= i\hbar L_y && (7)
\end{aligned}$$

For a vector wave function, the rotation is generated by J_i rather than L_i but because the effects of rotations are the same, the J_i must have the same commutation relations, so that

$$\begin{aligned}
[J_x, J_y] &= i\hbar J_z && (8)\\
[J_y, J_z] &= i\hbar J_x && (9)\\
[J_z, J_x] &= i\hbar J_y && (10)
\end{aligned}$$

We can do the same analysis on J as we did above with L to define the raising and lowering operators

$$J_\pm \equiv J_x \pm iJ_y \qquad (11)$$

and get the same eigenvalue relations

$$J_\pm |jm\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\,|j, m\pm 1\rangle \qquad (12)$$

The three components of **J** are then J_z and

$$\begin{aligned}
J_x &= \frac{1}{2}(J_+ + J_-) && (13)\\
J_y &= \frac{1}{2i}(J_+ - J_-) && (14)
\end{aligned}$$

Using these three equations, we can generate the matrix elements of the components of **J** in the orthonormal basis $|jm\rangle$ (that is, the basis consisting of eigenfunctions with total angular momentum number j and J_z number m). These matrix elements are

$$\begin{aligned}
\left\langle j'm' \left| J_x \right| jm \right\rangle &= \frac{1}{2}\left\langle j'm' \left| J_+ + J_- \right| jm \right\rangle && (15)\\
&= \frac{\hbar}{2}\sqrt{(j-m)(j+m+1)}\left\langle j'm' \left| j, m+1 \right.\right\rangle + && (16)\\
&\frac{\hbar}{2}\sqrt{(j+m)(j-m+1)}\left\langle j'm' \left| j, m-1 \right.\right\rangle && (17)\\
&= \frac{\hbar}{2}\left[\sqrt{(j-m)(j+m+1)}\delta_{j'j}\delta_{m',m+1} + \sqrt{(j+m)(j-m+1)}\delta_{j'j}\delta_{m',m-1}\right] && (18)
\end{aligned}$$

$$\langle j'm'|J_y|jm\rangle = \frac{1}{2i}\langle j'm'|J_+ - J_-|jm\rangle \tag{19}$$

$$= \frac{\hbar}{2i}\sqrt{(j-m)(j+m+1)}\langle j'm'|j,m+1\rangle - \tag{20}$$

$$\frac{\hbar}{2i}\sqrt{(j+m)(j-m+1)}\langle j'm'|j,m-1\rangle \tag{21}$$

$$= \frac{\hbar}{2i}\left[\sqrt{(j-m)(j+m+1)}\delta_{j'j}\delta_{m',m+1} - \sqrt{(j+m)(j-m+1)}\delta_{j'j}\delta_{m',m-1}\right] \tag{22}$$

$$\langle j'm'|J_z|jm\rangle = m\hbar\delta_{j'j}\delta_{m',m} \tag{23}$$

The full matrix for each component J_i is actually infinite-dimensional, since j can be any half-integer from 0 up to infinity. However, the sub-matrix for each value of j is completely orthogonal to all other sub-matrices with different j values, so the complete matrix for each J_i is block-diagonal. Shankar gives the matrices for J_x and J_y up to $j=1$ in his equations 12.5.23 and 12.5.24. This means that the commutation relations 9 should be obeyed for each set of sub-matrices corresponding to a particular j value.

For $j=\frac{1}{2}$ we have for the 3 sub-matrices (we can copy these from Shankar or use the above formulas to work them out). The values of m are $+\frac{1}{2}$ and $-\frac{1}{2}$ in that order, from top to bottom and left to right.

$$J_x^{(1/2)} = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{24}$$

$$J_y^{(1/2)} = \frac{\hbar}{2i}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{25}$$

$$= \frac{i\hbar}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{26}$$

$$\left[J_x^{(1/2)}, J_y^{(1/2)}\right] = \frac{i\hbar^2}{4}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) \tag{27}$$

$$\frac{i\hbar^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{28}$$

$$= i\hbar J_z^{(1/2)} \tag{29}$$

For $j=1$ we have for the 3 sub-matrices

$$J_x^{(1)} = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{30}$$

$$J_y^{(1)} = \frac{i\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{31}$$

$$\left[J_x^{(1)}, J_y^{(1)}\right] = \frac{i\hbar^2}{2} \left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right) \tag{32}$$

$$= i\hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{33}$$

$$= i\hbar J_z^{(1)} \tag{34}$$

For $j = \frac{3}{2}$ we need to work out the matrices from the formulas above for the matrix elements. Ordering the values of $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}$ from left to right (columns) and top to bottom (rows), we get

$$J_x^{(3/2)} = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \tag{35}$$

$$J_y^{(3/2)} = \frac{\hbar}{2i}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{bmatrix} \tag{36}$$

$$= \frac{i\hbar}{2}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \tag{37}$$

$$\left[J_x^{(3/2)}, J_y^{(3/2)}\right] = \frac{i\hbar^2}{4}\left(\begin{bmatrix} 3 & 0 & -2\sqrt{3} & 0 \\ 0 & 1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & -1 & 0 \\ 0 & 2\sqrt{3} & 0 & -3 \end{bmatrix} - \begin{bmatrix} -3 & 0 & -2\sqrt{3} & 0 \\ 0 & -1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & 1 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{bmatrix}\right) \tag{38}$$

$$= i\hbar^2\begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix} \tag{39}$$

$$= i\hbar J_z^{(3/2)} \tag{40}$$

Thus the commutation relation $[J_x, J_y] = i\hbar J_z$ is satisfied for these three sets of sub-matrices.

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