

## TOTAL ANGULAR MOMENTUM - MATRIX ELEMENTS AND COMMUTATION RELATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.2.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In Shankar's Chapter 12 treatment of the eigenvalues of the angular momentum operators  $L^2$  and  $L_z$ , he retraces much of what we've already covered as a result of working through Griffiths's book. He defines raising and lowering operators for angular momentum as

$$(0.1) \quad L_{\pm} \equiv L_x \pm iL_y$$

These operators can be used to discover the eigenvalues of  $L^2$  to be  $\ell(\ell+1)\hbar^2$ , where  $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$  and the eigenvalues of  $L_z$  are  $m\hbar$  where  $m$  ranges from  $-\ell$  to  $+\ell$  in integer steps. The eigenvalues of  $L_{\pm}$  can also be found to satisfy

$$(0.2) \quad L_{\pm} |\ell m\rangle = \hbar \sqrt{(\ell \mp m)(\ell \pm m + 1)} |\ell, m \pm 1\rangle$$

When dealing with vector wave functions (as opposed to scalar ones) in two dimensions, we found that a quantity  $J_z$  is the generator of infinitesimal rotations about the  $z$  axis, where

$$(0.3) \quad J_z = L_z + S_z$$

and the operator producing the rotation by  $\epsilon_z$  is

$$(0.4) \quad U[R(\epsilon_z \hat{z})] = I - \frac{i\epsilon_z}{\hbar} J_z$$

For a scalar wave function in three dimensions, we found that the properties of two successive rotations by  $\epsilon_x$  about the  $x$  axis and  $\epsilon_y$  about the  $y$  axis led to the commutations

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$$(0.5) \quad [L_x, L_y] = i\hbar L_z$$

$$(0.6) \quad [L_y, L_z] = i\hbar L_x$$

$$(0.7) \quad [L_z, L_x] = i\hbar L_y$$

For a vector wave function, the rotation is generated by  $J_i$  rather than  $L_i$  but because the effects of rotations are the same, the  $J_i$  must have the same commutation relations, so that

$$(0.8) \quad [J_x, J_y] = i\hbar J_z$$

$$(0.9) \quad [J_y, J_z] = i\hbar J_x$$

$$(0.10) \quad [J_z, J_x] = i\hbar J_y$$

We can do the same analysis on  $J$  as we did above with  $L$  to define the raising and lowering operators

$$(0.11) \quad J_{\pm} \equiv J_x \pm iJ_y$$

and get the same eigenvalue relations

$$(0.12) \quad J_{\pm} |jm\rangle = \hbar \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle$$

The three components of  $\mathbf{J}$  are then  $J_z$  and

$$(0.13) \quad J_x = \frac{1}{2}(J_+ + J_-)$$

$$(0.14) \quad J_y = \frac{1}{2i}(J_+ - J_-)$$

Using these three equations, we can generate the matrix elements of the components of  $\mathbf{J}$  in the orthonormal basis  $|jm\rangle$  (that is, the basis consisting of eigenfunctions with total angular momentum number  $j$  and  $J_z$  number  $m$ ). These matrix elements are

(0.15)

$$\langle j'm' | J_x | jm \rangle = \frac{1}{2} \langle j'm' | J_+ + J_- | jm \rangle$$

(0.16)

$$= \frac{\hbar}{2} \sqrt{(j-m)(j+m+1)} \langle j'm' | j, m+1 \rangle +$$

(0.17)

$$\frac{\hbar}{2} \sqrt{(j+m)(j-m+1)} \langle j'm' | j, m-1 \rangle$$

(0.18)

$$= \frac{\hbar}{2} \left[ \sqrt{(j-m)(j+m+1)} \delta_{j'j} \delta_{m',m+1} + \sqrt{(j+m)(j-m+1)} \delta_{j'j} \delta_{m',m-1} \right]$$

(0.19)

$$\langle j'm' | J_y | jm \rangle = \frac{1}{2i} \langle j'm' | J_+ - J_- | jm \rangle$$

(0.20)

$$= \frac{\hbar}{2i} \sqrt{(j-m)(j+m+1)} \langle j'm' | j, m+1 \rangle -$$

(0.21)

$$\frac{\hbar}{2i} \sqrt{(j+m)(j-m+1)} \langle j'm' | j, m-1 \rangle$$

(0.22)

$$= \frac{\hbar}{2i} \left[ \sqrt{(j-m)(j+m+1)} \delta_{j'j} \delta_{m',m+1} - \sqrt{(j+m)(j-m+1)} \delta_{j'j} \delta_{m',m-1} \right]$$

(0.23)

$$\langle j'm' | J_z | jm \rangle = m\hbar \delta_{j'j} \delta_{m',m}$$

The full matrix for each component  $J_i$  is actually infinite-dimensional, since  $j$  can be any half-integer from 0 up to infinity. However, the sub-matrix for each value of  $j$  is completely orthogonal to all other sub-matrices with different  $j$  values, so the complete matrix for each  $J_i$  is block-diagonal. Shankar gives the matrices for  $J_x$  and  $J_y$  up to  $j = 1$  in his equations 12.5.23 and 12.5.24. This means that the commutation relations 0.9 should be obeyed for each set of sub-matrices corresponding to a particular  $j$  value.

For  $j = \frac{1}{2}$  we have for the 3 sub-matrices (we can copy these from Shankar or use the above formulas to work them out). The values of  $m$  are  $+\frac{1}{2}$  and  $-\frac{1}{2}$  in that order, from top to bottom and left to right.

$$(0.24) \quad J_x^{(1/2)} = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(0.25) \quad J_y^{(1/2)} = \frac{\hbar}{2i} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$(0.26) \quad = \frac{i\hbar}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(0.27) \quad [J_x^{(1/2)}, J_y^{(1/2)}] = \frac{i\hbar^2}{4} \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$(0.28) \quad = \frac{i\hbar^2}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$(0.29) \quad = i\hbar J_z^{(1/2)}$$

For  $j = 1$  we have for the 3 sub-matrices

$$(0.30) \quad J_x^{(1)} = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(0.31) \quad J_y^{(1)} = \frac{i\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(0.32) \quad [J_x^{(1)}, J_y^{(1)}] = \frac{i\hbar^2}{2} \left( \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right)$$

$$(0.33) \quad = i\hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$(0.34) \quad = i\hbar J_z^{(1)}$$

For  $j = \frac{3}{2}$  we need to work out the matrices from the formulas above for the matrix elements. Ordering the values of  $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}$  from left to right (columns) and top to bottom (rows), we get

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(0.35)

$$J_x^{(3/2)} = \frac{\hbar}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

(0.36)

$$J_y^{(3/2)} = \frac{\hbar}{2i} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{bmatrix}$$

(0.37)

$$= \frac{i\hbar}{2} \begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

(0.38)

$$[J_x^{(3/2)}, J_y^{(3/2)}] = \frac{i\hbar^2}{4} \left( \begin{bmatrix} 3 & 0 & -2\sqrt{3} & 0 \\ 0 & 1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & -1 & 0 \\ 0 & 2\sqrt{3} & 0 & -3 \end{bmatrix} - \begin{bmatrix} -3 & 0 & -2\sqrt{3} & 0 \\ 0 & -1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & 1 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{bmatrix} \right)$$

(0.39)

$$= i\hbar^2 \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix}$$

(0.40)

$$= i\hbar J_z^{(3/2)}$$

Thus the commutation relation  $[J_x, J_y] = i\hbar J_z$  is satisfied for these three sets of sub-matrices.

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