TOTAL ANGULAR MOMENTUM - MATRIX ELEMENTS AND COMMUTATION RELATIONS

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.2.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

In Shankar's Chapter 12 treatment of the eigenvalues of the angular momentum operators L^2 and L_z, he retraces much of what we've already covered as a result of working through Griffiths's book. He defines raising and lowering operators for angular momentum as

$$L_{\pm}\equiv L_x\pm iL_y \tag{1}$$

These operators can be used to discover the eigenvalues of L^2 to be $\ell(\ell+1)\hbar^2$, where $\ell=0,\frac{1}{2},1,\frac{3}{2},\ldots$ and the eigenvalues of L_z are $m\hbar$ where m ranges from $-\ell$ to $+\ell$ in integer steps. The eigenvalues of $L_{\pm}$ can also be found to satisfy

$$L_{\pm}|\ell m\rangle=\hbar\sqrt{(\ell\mp m)(\ell\pm m+1)}\,|\ell,m\pm 1\rangle \tag{2}$$

When dealing with vector wave functions (as opposed to scalar ones) in two dimensions, we found that a quantity J_z is the generator of infinitesimal rotations about the z axis, where

$$J_z=L_z+S_z \tag{3}$$

and the operator producing the rotation by ε_z is

$$U[R(\varepsilon_z\hat{\mathbf{z}})]=I-\frac{i\varepsilon_z}{\hbar}J_z \tag{4}$$

For a scalar wave function in three dimensions, we found that the properties of two successive rotations by ε_x about the x axis and ε_y about the y axis led to the commutations

$$
\begin{aligned}
&(5) \qquad [L_x, L_y] = i\hbar L_z \\
&(6) \qquad [L_y, L_z] = i\hbar L_x \\
&(7) \qquad [L_z, L_x] = i\hbar L_y
\end{aligned}
$$

For a vector wave function, the rotation is generated by J_i rather than L_i but because the effects of rotations are the same, the J_i must have the same commutation relations, so that

$$
\begin{aligned}
&(8) \qquad [J_x, J_y] = i\hbar J_z \\
&(9) \qquad [J_y, J_z] = i\hbar J_x \\
&(10) \qquad [J_z, J_x] = i\hbar J_y
\end{aligned}
$$

We can do the same analysis on J as we did above with L to define the raising and lowering operators

(11) $$J_{\pm} \equiv J_x \pm iJ_y$$

and get the same eigenvalue relations

(12) $$J_{\pm} |jm\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle$$

The three components of $\mathbf{J}$ are then J_z and

$$
\begin{aligned}
&(13) \qquad J_x = \frac{1}{2}(J_+ + J_-) \\
&(14) \qquad J_y = \frac{1}{2i}(J_+ - J_-)
\end{aligned}
$$

Using these three equations, we can generate the matrix elements of the components of $\mathbf{J}$ in the orthonormal basis $|jm\rangle$ (that is, the basis consisting of eigenfunctions with total angular momentum number j and J_z number m). These matrix elements are

(15)
$$\left\langle j'm'\left|J_x\right|jm\right\rangle = \frac{1}{2}\left\langle j'm'\left|J_+ + J_-\right|jm\right\rangle$$
(16)
$$= \frac{\hbar}{2}\sqrt{(j-m)(j+m+1)}\left\langle j'm'\left|j,m+1\right.\right\rangle +$$
(17)
$$\frac{\hbar}{2}\sqrt{(j+m)(j-m+1)}\left\langle j'm'\left|j,m-1\right.\right\rangle$$
(18)
$$= \frac{\hbar}{2}\left[\sqrt{(j-m)(j+m+1)}\delta_{j'j}\delta_{m',m+1} + \sqrt{(j+m)(j-m+1)}\delta_{j'j}\delta_{m',m-1}\right]$$

(19)
$$\left\langle j'm'\left|J_y\right|jm\right\rangle = \frac{1}{2i}\left\langle j'm'\left|J_+ - J_-\right|jm\right\rangle$$
(20)
$$= \frac{\hbar}{2i}\sqrt{(j-m)(j+m+1)}\left\langle j'm'\left|j,m+1\right.\right\rangle -$$
(21)
$$\frac{\hbar}{2i}\sqrt{(j+m)(j-m+1)}\left\langle j'm'\left|j,m-1\right.\right\rangle$$
(22)
$$= \frac{\hbar}{2i}\left[\sqrt{(j-m)(j+m+1)}\delta_{j'j}\delta_{m',m+1} - \sqrt{(j+m)(j-m+1)}\delta_{j'j}\delta_{m',m-1}\right]$$

(23)
$$\left\langle j'm'\left|J_z\right|jm\right\rangle = m\hbar\delta_{j'j}\delta_{m',m}$$

The full matrix for each component J_i is actually infinite-dimensional, since j can be any half-integer from 0 up to infinity. However, the sub-matrix for each value of j is completely orthogonal to all other sub-matrices with different j values, so the complete matrix for each J_i is block-diagonal. Shankar gives the matrices for J_x and J_y up to $j=1$ in his equations 12.5.23 and 12.5.24. This means that the commutation relations 9 should be obeyed for each set of sub-matrices corresponding to a particular j value.

For $j=\frac{1}{2}$ we have for the 3 sub-matrices (we can copy these from Shankar or use the above formulas to work them out). The values of m are $+\frac{1}{2}$ and $-\frac{1}{2}$ in that order, from top to bottom and left to right.

$$J_x^{(1/2)} = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{24}$$

$$J_y^{(1/2)} = \frac{\hbar}{2i}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{25}$$

$$= \frac{i\hbar}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{26}$$

$$\left[J_x^{(1/2)}, J_y^{(1/2)}\right] = \frac{i\hbar^2}{4}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) \tag{27}$$

$$\frac{i\hbar^2}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{28}$$

$$= i\hbar J_z^{(1/2)} \tag{29}$$

For $j = 1$ we have for the 3 sub-matrices

$$J_x^{(1)} = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{30}$$

$$J_y^{(1)} = \frac{i\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{31}$$

$$\left[J_x^{(1)}, J_y^{(1)}\right] = \frac{i\hbar^2}{2}\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right) \tag{32}$$

$$= i\hbar^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{33}$$

$$= i\hbar J_z^{(1)} \tag{34}$$

For $j = \frac{3}{2}$ we need to work out the matrices from the formulas above for the matrix elements. Ordering the values of $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}$ from left to right (columns) and top to bottom (rows), we get

(35)

$$J_x^{(3/2)} = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

(36)

$$J_y^{(3/2)} = \frac{\hbar}{2i}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{bmatrix}$$

(37)

$$= \frac{i\hbar}{2}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

(38)

$$\left[J_x^{(3/2)}, J_y^{(3/2)}\right] = \frac{i\hbar^2}{4}\left(\begin{bmatrix} 3 & 0 & -2\sqrt{3} & 0 \\ 0 & 1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & -1 & 0 \\ 0 & 2\sqrt{3} & 0 & -3 \end{bmatrix} - \begin{bmatrix} -3 & 0 & -2\sqrt{3} & 0 \\ 0 & -1 & 0 & -2\sqrt{3} \\ 2\sqrt{3} & 0 & 1 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{bmatrix}\right)$$

(39)

$$= i\hbar^2\begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix}$$

(40)

$$= i\hbar J_z^{(3/2)}$$

Thus the commutation relation $[J_x, J_y] = i\hbar J_z$ is satisfied for these three sets of sub-matrices.

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