ANGULAR MOMENTUM IN 3-D EXPECTATION VALUES AND UNCERTAINTY PRINCIPLE

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.3.

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For 3-d angular momentum, we've seen that the components J_x and J_y can be written in terms of raising and lowering operators

$$J_{\pm} \equiv J_x \pm iJ_y \tag{1}$$

In the basis of eigenvectors of J^2 and J_z (that is, the states $|jm\rangle$) the raising and lowering operators have the following effects:

$$J_{\pm}|jm\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}\,|j, m \pm 1\rangle \tag{2}$$

We can use these relations to construct the matrix elements of J_x and J_y in this basis. We can also use these relations to work out expectation values and uncertainties for the angular momentum components in this basis.

First, since diagonals of both the J_x and J_y matrices have only zero elements,

$$\begin{aligned}
\langle J_x\rangle &= \langle jm|J_x|jm\rangle = 0 && (3)\\
\langle J_y\rangle &= \langle jm|J_y|jm\rangle = 0 && (4)
\end{aligned}$$

To work out $\langle J_x^2\rangle$ and $\langle J_y^2\rangle$, we can write these operators in terms of the raising and lowering operators:

$$\begin{aligned}
J_x &= \frac{1}{2}(J_+ + J_-) && (5)\\
J_y &= \frac{1}{2i}(J_+ - J_-) && (6)
\end{aligned}$$

We can then use the fact that the basis states are orthonormal, so that

$$\langle j'm'|jm\rangle = \delta_{j'j}\delta_{m'm} \tag{7}$$

The required squares are

$$J_x^2 = \frac{1}{4}\left(J_+^2 + J_+J_- + J_-J_+ + J_-^2\right) \tag{8}$$

$$J_y^2 = -\frac{1}{4}\left(J_+^2 - J_+J_- - J_-J_+ + J_-^2\right) \tag{9}$$

$$= \frac{1}{4}\left(-J_+^2 + J_+J_- + J_-J_+ - J_-^2\right) \tag{10}$$

The diagonal matrix elements $\left\langle jm\left|J_x^2\right|jm\right\rangle$ and $\left\langle jm\left|J_y^2\right|jm\right\rangle$ will get non-zero contributions only from those terms that leave j and m unchanged when operating on $|jm\rangle$. This means that only the terms that contain an equal number of J_+ and J_- terms will contribute. We therefore have

$$\left\langle jm\left|J_x^2\right|jm\right\rangle = \frac{1}{4}\left\langle jm\left|J_+J_- + J_-J_+\right|jm\right\rangle \tag{11}$$

$$= \frac{\hbar}{4}\sqrt{(j+m)(j-m+1)}\left\langle jm\left|J_+\right|j,m-1\right\rangle + \tag{12}$$

$$\frac{\hbar}{4}\sqrt{(j-m)(j+m+1)}\left\langle jm\left|J_-\right|j,m+1\right\rangle \tag{13}$$

$$= \frac{\hbar^2}{4}\sqrt{(j+m)(j-m+1)}\sqrt{(j-m+1)(j+m)} + \tag{14}$$

$$\frac{\hbar^2}{4}\sqrt{(j-m)(j+m+1)}\sqrt{(j+m+1)(j-m)} \tag{15}$$

$$= \frac{\hbar^2}{4}\left((j+m)(j-m+1) + (j-m)(j+m+1)\right) \tag{16}$$

$$= \frac{\hbar^2}{4}\left(j^2 - m^2 + j + m + j^2 - m^2 + j - m\right) \tag{17}$$

$$= \frac{\hbar^2}{2}\left(j(j+1) - m^2\right) \tag{18}$$

From 10 we see that the only terms that contribute to $\left\langle jm\left|J_y^2\right|jm\right\rangle$ are the same as the corresponding terms in $\left\langle jm\left|J_x^2\right|jm\right\rangle$, so the result is the same:

$$\left\langle jm\left|J_y^2\right|jm\right\rangle = \frac{\hbar^2}{2}\left(j(j+1) - m^2\right) \tag{19}$$

We can check that J_x and J_y satisfy the uncertainty principle, as derived by Shankar. That is, we want to verify that

$$\Delta J_x \cdot \Delta J_y \geq \left|\left\langle jm\left|\left(J_x - \langle J_x\rangle\right)\left(J_y - \left\langle J_y\right\rangle\right)\right|jm\right\rangle\right| \tag{20}$$

On the LHS

$$\Delta J_x = \sqrt{\langle J_x^2 \rangle - \langle J_x \rangle^2} \tag{21}$$

$$= \sqrt{\langle J_x^2 \rangle} \tag{22}$$

$$= \sqrt{\frac{\hbar^2}{2}\left(j(j+1) - m^2\right)} \tag{23}$$

$$\Delta J_y = \sqrt{\frac{\hbar^2}{2}\left(j(j+1) - m^2\right)} \tag{24}$$

$$\Delta J_x \cdot \Delta J_y = \frac{\hbar^2}{2}\left(j(j+1) - m^2\right) \tag{25}$$

On the RHS

$$\left|\left\langle jm \left| \left(J_x - \langle J_x \rangle\right)\left(J_y - \langle J_y \rangle\right) \right| jm \right\rangle\right| = \left|\left\langle jm \left| J_x J_y \right| jm \right\rangle\right| \tag{26}$$

Using the same technique as that above for deriving $\left\langle jm \left| J_x^2 \right| jm \right\rangle$ we have

$$\left\langle jm \left| J_x J_y \right| jm \right\rangle = \frac{1}{4i}\left\langle jm \left| \left(J_+ + J_-\right)\left(J_+ - J_-\right) \right| jm \right\rangle \tag{27}$$

$$= \frac{1}{4i}\left\langle jm \left| J_- J_+ - J_+ J_- \right| jm \right\rangle \tag{28}$$

$$= \frac{\hbar^2}{4i}\left((j-m)(j+m+1) - (j+m)(j-m+1)\right) \tag{29}$$

$$= -\frac{\hbar^2 m}{2i} \tag{30}$$

We therefore need to verify that

$$j(j+1) - m^2 \geq |m| \tag{31}$$

for all allowed values of m. We know that $-j \leq m \leq +j$, so

$$j(j+1) - m^2 \geq j^2 + j - j^2 = j \geq |m| \tag{32}$$

Thus the inequality is indeed satisfied.
In the case $|m| = j$ we have

$$j(j+1) - j^2 = j = |m| \tag{33}$$

so the inequality saturates (becomes an equality) in that case.