ROTATIONS IN 3-D EULER ANGLES

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Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 12, Exercise 12.5.7.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

Any 3-d rotation can be expressed in terms of the Euler angles. These angles specify a sequence of three successive rotations about the rectangular axes. There are various definitions of Euler angles involving different sets of rotations, but the set used by Shankar in this problem consists of (1) a rotation by γ about the z axis, followed by (2) a rotation by β about the y axis and concluding with (3) a rotation by α about the z axis. The proof that any rotation can be expressed this way would take us too far afield at this point, so we'll just accept this for now.

We can see how these work in quantum mechanics by considering the special case of $j=1$, for which we derived the formula for a finite rotation here. A state $|\psi\rangle$ is transformed by a rotation θ according to

$$\left|\psi'\right\rangle = D^{(1)}[R]\,|\psi\rangle \tag{1}$$

$$= \left[I^{(1)} + \frac{\left(\hat{\theta}\cdot\mathbf{J}^{(1)}\right)^2}{\hbar^2}(\cos\theta-1) - \frac{i\hat{\theta}\cdot\mathbf{J}^{(1)}}{\hbar}\sin\theta\right]|\psi\rangle \tag{2}$$

For our purposes below, we'll need the three components of $\mathbf{J}^{(1)}$, which can be copied from Shankar's equations 12.5.23 and 12.5.24:

$$J_x^{(1)} = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{3}$$

$$J_y^{(1)} = \frac{i\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4}$$

$$\left(J_y^{(1)}\right)^2 = \frac{\hbar^2}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{5}$$

$$J_z^{(1)} = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{6}$$

$$\left(J_z^{(1)}\right)^2 = \hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Evaluating 2 for the three Euler rotations, we have

$$
\begin{aligned}
D_\gamma &= I^{(1)} + \frac{\left(J_z^{(1)}\right)^2}{\hbar^2}(\cos\gamma - 1) - \frac{iJ_z^{(1)}}{\hbar}\sin\gamma && (8)\\
&= \begin{bmatrix} \cos\gamma - i\sin\gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\gamma + i\sin\gamma \end{bmatrix} && (9)\\
&= \begin{bmatrix} e^{-i\gamma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\gamma} \end{bmatrix} && (10)\\
D_\beta &= I^{(1)} + \frac{\left(J_y^{(1)}\right)^2}{\hbar^2}(\cos\beta - 1) - \frac{iJ_y^{(1)}}{\hbar}\sin\beta && (11)\\
&= \frac{1}{2}\begin{bmatrix} 1+\cos\beta & -\sqrt{2}\sin\beta & 1-\cos\beta \\ \sqrt{2}\sin\beta & 2\cos\beta & -\sqrt{2}\sin\beta \\ 1-\cos\beta & \sqrt{2}\sin\beta & 1+\cos\beta \end{bmatrix} && (12)\\
D_\alpha &= I^{(1)} + \frac{\left(J_z^{(1)}\right)^2}{\hbar^2}(\cos\alpha - 1) - \frac{iJ_z^{(1)}}{\hbar}\sin\alpha && (13)\\
&= \begin{bmatrix} \cos\alpha - i\sin\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\alpha + i\sin\alpha \end{bmatrix} && (14)\\
&= \begin{bmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{bmatrix} && (15)
\end{aligned}
$$

The complete rotation is the product of the three matrices:

$$
\begin{aligned}
D_{total} &= D_\alpha D_\beta D_\gamma && (16)\\
&= D_\alpha \frac{1}{2}\begin{bmatrix} (1+\cos\beta)e^{-i\gamma} & -\sqrt{2}\sin\beta & (1-\cos\beta)e^{i\gamma} \\ \sqrt{2}\sin\beta e^{-i\gamma} & 2\cos\beta & -\sqrt{2}\sin\beta e^{i\gamma} \\ (1-\cos\beta)e^{-i\gamma} & \sqrt{2}\sin\beta & (1+\cos\beta)e^{i\gamma} \end{bmatrix} && (17)\\
&= \frac{1}{2}\begin{bmatrix} (1+\cos\beta)e^{-i\gamma}e^{-i\alpha} & -\sqrt{2}\sin\beta e^{-i\alpha} & (1-\cos\beta)e^{i\gamma}e^{-i\alpha} \\ \sqrt{2}\sin\beta e^{-i\gamma} & 2\cos\beta & -\sqrt{2}\sin\beta e^{i\gamma} \\ (1-\cos\beta)e^{-i\gamma}e^{i\alpha} & \sqrt{2}\sin\beta e^{i\alpha} & (1+\cos\beta)e^{i\gamma}e^{i\alpha} \end{bmatrix} && (18)
\end{aligned}
$$

In the $|jm\rangle$ basis, the state $|11\rangle$ is represented by

$$|11\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

Applying the rotation 18 to this state, we get

$$\begin{aligned}
|11'\rangle &= D_{total}\,|11\rangle &(20)\\
&= \frac{1}{2}\begin{bmatrix} (1+\cos\beta)\,e^{-i\gamma}e^{-i\alpha} & -\sqrt{2}\sin\beta e^{-i\alpha} & (1-\cos\beta)\,e^{i\gamma}e^{-i\alpha} \\ \sqrt{2}\sin\beta e^{-i\gamma} & 2\cos\beta & -\sqrt{2}\sin\beta e^{i\gamma} \\ (1-\cos\beta)\,e^{-i\gamma}e^{i\alpha} & \sqrt{2}\sin\beta e^{i\alpha} & (1+\cos\beta)\,e^{i\gamma}e^{i\alpha} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &(21)\\
&= \frac{1}{2}\begin{bmatrix} (1+\cos\beta)\,e^{-i\gamma}e^{-i\alpha} \\ \sqrt{2}\sin\beta e^{-i\gamma} \\ (1-\cos\beta)\,e^{-i\gamma}e^{i\alpha} \end{bmatrix} &(22)\\
&= \frac{e^{-i\gamma}}{2}\begin{bmatrix} (1+\cos\beta)\,e^{-i\alpha} \\ \sqrt{2}\sin\beta \\ (1-\cos\beta)\,e^{i\alpha} \end{bmatrix} &(23)
\end{aligned}$$

We can work out the average values of the components of **J** in the rotated state in the same way as in the previous problem, by using 3, 4 and 6. Note that γ disappears from the matrix elements as it enters only in an overall phase factor. We get (I used Maple to do the tedious matrix multiplications):

$$\langle J_x\rangle = \left\langle 11'\right| J_x \left| 11'\right\rangle \tag{24}$$

$$= \frac{\hbar}{4\sqrt{2}} \left[\begin{array}{ccc} (1+\cos\beta)\,e^{i\alpha} & \sqrt{2}\sin\beta & (1-\cos\beta)\,e^{-i\alpha} \end{array}\right] \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} (1+\cos\beta)\,e^{-i\alpha} \\ \sqrt{2}\sin\beta \\ (1-\cos\beta)\,e^{i\alpha} \end{array}\right] \tag{25}$$

$$= \hbar\sin\beta\cos\alpha \tag{26}$$

$$\langle J_y\rangle = \left\langle 11'\right| J_y \left| 11'\right\rangle \tag{27}$$

$$= \frac{i\hbar}{4\sqrt{2}} \left[\begin{array}{ccc} (1+\cos\beta)\,e^{i\alpha} & \sqrt{2}\sin\beta & (1-\cos\beta)\,e^{-i\alpha} \end{array}\right] \left[\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} (1+\cos\beta)\,e^{-i\alpha} \\ \sqrt{2}\sin\beta \\ (1-\cos\beta)\,e^{i\alpha} \end{array}\right] \tag{28}$$

$$= \hbar\sin\beta\sin\alpha \tag{29}$$

$$\langle J_z\rangle = \left\langle 11'\right| J_z \left| 11'\right\rangle \tag{30}$$

$$= \frac{\hbar}{4} \left[\begin{array}{ccc} (1+\cos\beta)\,e^{i\alpha} & \sqrt{2}\sin\beta & (1-\cos\beta)\,e^{-i\alpha} \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array}\right] \left[\begin{array}{c} (1+\cos\beta)\,e^{-i\alpha} \\ \sqrt{2}\sin\beta \\ (1-\cos\beta)\,e^{i\alpha} \end{array}\right] \tag{31}$$

$$= \hbar\cos\beta \tag{32}$$

Going back to 23, we can confirm our earlier result that it is impossible to rotate the state $|11\rangle$ into just $|10\rangle$. To do so, the state in 23 would have to be a multiple of $\left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]$ which could happen only if both $1+\cos\beta$ and $1-\cos\beta$ were zero, which is impossible.

However, if we take $\beta = \pi$, then the rotated state 23 becomes

$$\left|11'\right\rangle = e^{i(\alpha-\gamma)} \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] = e^{i(\alpha-\gamma)}\,|1,-1\rangle \tag{33}$$

so apart from a phase factor, it is possible to rotate one eigenstate of J_z into another.

The only values of β that produce zero elements in 23 are $\beta = 0$ and $\beta = \pi$, (the values of α and γ produce only phase factors), so for any other value of β, all three elements of 23 are non-zero. Thus a general rotation from any starting state can always be made to produce a rotated state containing all three eigenstates of J_z: $|11\rangle$, $|10\rangle$ and $|1,-1\rangle$.